HEAVISIDE'S ELECTRICAL CIRCUIT THEORY

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WITH AN INTRODUCTION BY
PROFESSOR M. I. PUPIN

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PREFACE

The importance of Heaviside's contributions to electrical theory is now generally recognized and appreciated. His teachings nevertheless are available to only a comparatively few; to the many engineers and physicists who could profit much by it, the work of Heaviside is more or less a sealed book. This may be accounted for largely as due to the novel and original mathematical processes he has introduced and applied with such extraordinary skill in the solution of many problems; and also to some extent to the lack of any attempt to correlate and present his teachings in a systematic manner suitable for one approaching the subject for the first time.

Heaviside was not only a great mathematician, he was also a great physicist; and it is the knowledge of the physics of the problems which guided him correctly in many instances to the development of suitable mathematical processes. He concerned himself little with formal proofs or rigorous demonstrations. As he remarked: "In working out physical problems, there should be, in the first place, no pretense to rigorous formalism. The physics will guide the physicist along somehow to useful and important results, by the constant union of physical and geometrical or analytical ideas."

There is really nothing intrinsically difficult about Heaviside's mathematical processes. Once the underlying principles of the operational methods are understood, and for that, a knowledge of comparatively elementary mathematics only, is required, a way is immediately opened to the student for the utilization of a considerable part of Heaviside's work, particularly in so far as it relates to engineering problems. As an illustration, we may refer to the now celebrated Expansion Theorem. Electrical engineers could be easily taught its meaning and use in the solution of problems, and there is no reason why it should not have by this time found its way into engineering text-books.

This book is an attempt to present some of Heaviside's work in a manner to appeal to the engineer. It is put forth in the hope

that it will serve as a suitable introduction acquainting the student with the principles of operational calculus and their applications to engineering problems. Great care was taken in the choice of material and its arrangement with a view to unfolding the subject in a gradual way, and at the same time emphasizing throughout, the utility of these mathematical processes in facilitating the solution of engineering problems. All the way through, illustrations are introduced at every step in the development of the subject to make it more readily comprehensible, and at the same time serving to emphasize the utility of the mathematical methods. The Expansion Theorem is stressed throughout, showing its applicability to the solution of such varied problems as Transmission Lines, Electric Filter Circuits, Artificial Lines, and others.

In the choice of illustrations, I have not confined myself to Heaviside's writings, since he concerned himself mostly with problems relating to transmission lines and cables. It was rather my aim to show the wider applicability of Heaviside's mathematical processes, and for this reason, I sought to include varied problems. It is hoped that the discussions of some of the problems given here, such as filter circuits, artificial lines, and others, have some intrinsic merit of their own, and will be of interest to engineers.

The author wishes to acknowledge his deep obligations to Heaviside for the benefits derived from the study of his writings. This little book is put forward as a personal tribute to him and also with the expectation that it may serve as a stimulus for others to study his works.

The Author.

Washington, D. C.
August 2, 1928.

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INTRODUCTION

Doctor Louis Cohen's studies of Heaviside's work are well known from his publications, and they are highly appreciated.

Electrical engineering is to be congratulated upon the fact that Dr. Cohen in this book gives us a splendid summary of that part of Heaviside's mathematical analysis which bears upon the theory of the electrical circuit. It will be found that Dr. Cohen's presentation of this subject is so clear and simple that many of the difficulties which the student finds in Heaviside's original work have disappeared. Heaviside's work is epoch making and every electrical engineer should familiarize himself with it, to that extent, at least, which is given in the judiciously selected parts discussed in this excellent book.

M. I. Pupin.

Norfolk, Conn.
August 20, 1928.

HEAVISIDE'S ELECTRICAL CIRCUIT THEORY

CHAPTER I

OPERATIONAL CALCULUS

The Application of Operational Calculus to the Solution of Circuit Problems of Concentrated Inductance, Capacity, and Resistance.

Any investigation in electric-circuit theory, to be at all comprehensive, must necessarily concern itself with the study of the voltage and current distribution in the branches of any circuit network during the transient state as well as in the permanent state. The problems in electric-circuit theory relating to the steady-state condition do not generally offer any mathematical difficulties, even for the case of circuits of distributed inductance, capacity, and resistance; such problems are now fully discussed in engineering textbooks. The mathematical equipment of the engineering student is quite sufficient to enable him to follow intelligently any discussion relating to such problems. In the case, however, of the study of transient phenomena, problems arise which are much more difficult, and special mathematical methods must be provided to facilitate and, in some cases, to make at all possible a solution.

Heaviside was chiefly interested in telegraphic problems, dealing with matters relating to the propagation of electric impulses in conductors, and as such had to deal exclusively with transient effects. This led him to the development of the operational calculus, which he used with remarkable skill and brilliance in the solution of many problems which would otherwise have been either altogether impossible of solution or at least would have involved mathematics of great complexity. The mathematical methods which Heaviside evolved are, of course, applicable to the solution of problems in other branches of engineering, and

emphasis should be placed on the methods rather than on the specific problems used to illustrate the applicability of the methods.

To begin with, we shall consider a few simple problems utilizing direct operational methods to obtain the solutions, which may serve as an introduction to familiarize the student with the general concept of operational solutions.

Circuit of Inductance and Resistance.—Consider the case of the current rise in an inductive circuit. Suppose we have a circuit of inductance L and resistance R, and a steady voltage E applied, what is the currect in the circuit at any time after closing the circuit? We have, of course, the well-known differential equation for this circuit condition, which is as follows:

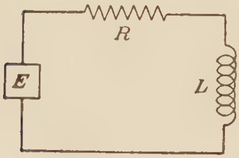

FIG. 1.

$$L\frac{di}{dt} + Ri = E. \tag{1}$$

The usual method for solving this equation, to obtain an expression for the current, is to assume a solution of the form:

$$i = A\epsilon^{\lambda t} + B. \tag{2}$$

which, on substitution, gives

$$(L\lambda + R)A\epsilon^{\lambda t} + BR = E. \tag{3}$$

Since this is to hold for all values of t, the following relations must be satisfied:

$$L\lambda + R = 0;\ BR = E,$$

which give

$$\lambda = -\frac{R}{L};\ B = \frac{E}{R}.$$

Hence,

$$i = A\epsilon^{-\frac{R}{L}t} + \frac{E}{R}.$$

To satisfy the condition that the current in the circuit is to have zero value when $t = 0$, the constant A must have the value $-\frac{E}{R}$. We finally arrive at the following equation:

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right), \tag{4}$$

the well-known expression for the current rise in an inductive circuit.

We shall now obtain the solution to the same problem by the operational method. Use the symbol p to designate differ-

entiation to time—that is, $p = d/dt$—and equation (1) takes this form:

$$(Lp + R)i = E,$$

and

$$i = \frac{E}{Lp + R}. \tag{5}$$

This is a symbolic solution, giving the current in terms of the constants of the circuit and the symbol p, which is to be algebrized, as Heaviside would say, to obtain the real solution. We proceed as follows: Write equation (5) in this form:

$$i = \frac{E}{Lp + R} = \frac{E}{Lp\left(1 + \frac{R}{Lp}\right)}, \tag{6}$$

by division, we have

$$i = \frac{1}{Lp}\left\{1 - \frac{R}{Lp} + \frac{R^2}{L^2p^2} - \frac{R^3}{L^3p^3} + \cdots\right\}E.$$

The operation on E by $1/p = p^{-1}$ signifies integration of E from 0 to t. This is a consequence of the following consideration: Given

$$pu = v,$$

then

$$u = p^{-1}v,$$

and

$$pu = pp^{-1}v = v.$$

Hence,

$$pp^{-1} = 1.$$

Thus, p^{-1} represents an operation on any quantity that if the operation by p be subsequently performed, the quantity is unaltered. An operation by p^{-1} is, therefore, equivalent to an integration. Operating by p^{-1} on a unity function (that is, one which has zero value for $t < 0$ and is equal to 1 for all values of $t > 0$), we get

$$\begin{aligned} \frac{1}{p} &= \int_0^t dt = t, \\ \frac{1}{p^2} &= \int_0^t t\,dt = \frac{t^2}{2!}, \\ &\cdots\cdots \\ \frac{1}{p^n} &= \int_0^t \frac{t^{n-1}}{(n-1)!}dt = \frac{t^n}{n!}. \end{aligned} \tag{8}$$

Using these integration values in (7), it transforms into the following:

$$i = \frac{E}{Lp}\left\{1 - \frac{R}{L}t + \frac{R^2}{L^2}\frac{t^2}{2!} - \frac{R^3}{L^3}\frac{t^3}{3!} + \cdots\right\}. \tag{9}$$

The bracket expression in (9) is the expanded form of $\epsilon^{-\frac{R}{L}t}$; hence,

$$i = \frac{E}{Lp}\epsilon^{-\frac{R}{L}t}. \tag{10}$$

Operating now on $\epsilon^{-\frac{R}{L}t}$ by $\frac{1}{p}$, that is, integrating the exponential factor, we obtain the complete solution for the current in the circuit as follows:

$$i = \frac{E}{L}\int_0^t \epsilon^{-\frac{R}{L}t}dt = -\frac{E}{R}\left[\epsilon^{-\frac{R}{L}t}\right]_0^t$$

$$= \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right). \tag{11}$$

which is the same as (4).

In this simple problem, the operational method does not appear to offer any particular advantage over the older method, but it serves to illustrate, by a simple example, the general process of the operational method. In more complex problems, the advantage of the operational method will become evident. We note, however, that in the development of the solution to the preceding problem, we have established the relation

$$\frac{1}{1 + \frac{R}{L}p} = \epsilon^{-\frac{R}{L}t},$$

or, more generally,

$$\frac{1}{1 + \frac{a}{p}} = \frac{p}{p + a} = \epsilon^{-at}. \tag{12}$$

Operating on unity function by $\frac{p}{p + a}$ yields ϵ^{-at}. This formula turns up frequently, and we shall have occasion to use it in connection with other problems. In fact, with the relation (12) given, which could have been established without regard to any particular problem, the solution to the preceding problem would have been much simplified.

Circuit of Capacity and Resistance.—Take the case of a circuit of capacity C and resistance R, and a steady voltage E applied, to obtain the expression for the charging current. The circuit equation for this case is

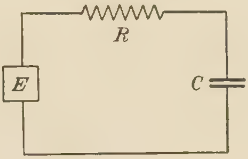

FIG. 2.

$$Ri + \frac{1}{C}\int idt = E,$$

symbolically

$$\left(R + \frac{1}{Cp}\right)i = E,$$

and

$$i = \frac{E}{R + \frac{1}{Cp}} = \frac{E}{R\left(1 + \frac{1}{RCp}\right)}. \tag{13}$$

This is of the same form as (12), $a = 1/RC$, and the solution can be written down at sight.

$$i = \frac{E}{R}\epsilon^{-\frac{1}{RC}t}. \tag{14}$$

Circuit of Inductance, Capacity, and Resistance.—As another example illustrating the applicability of direct operational method, we may consider a circuit comprising inductance, capacity, and resistance. The circuit equation expressed symbolically is as follows:

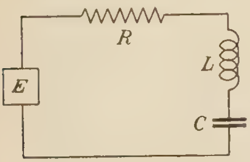

FIG. 3.

$$\left(Lp + R + \frac{1}{Cp}\right)i = E$$

and

$$i = \frac{E}{Lp + R + \frac{1}{Cp}}. \tag{15}$$

This equation can be put in the following form:

$$i = \frac{Ep}{L(p - p_1)(p - p_2)}, \tag{16}$$

where p_1 and p_2 are the roots of the quadratic equation

$$LCp^2 + RCp + 1 = 0,$$

that is,

$$\left.\begin{aligned} p_1 &= -\alpha + j\beta \\ p_2 &= -\alpha - j\beta \end{aligned}\right\} \tag{17}$$

where

$$\alpha = \frac{R}{2L}, \text{ and } \beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Equation (16) can be resolved by partial fractions into the following:

$$\begin{aligned} i &= \frac{Ep}{L(p_2 - p_1)}\left\{\frac{1}{p - p_2} - \frac{1}{p - p_1}\right\} \\ &= \frac{E}{L(p_2 - p_1)}\left\{\frac{1}{1 - \frac{p_2}{p}} - \frac{1}{1 - \frac{p_1}{p}}\right\}. \end{aligned} \tag{18}$$

By (12), each of the bracket terms operating on unity function is converted into an exponential term; thus:

$$\frac{1}{1 - \frac{p^2}{p}} = \epsilon^{p_2 t}; \quad \frac{1}{1 - \frac{p_1}{p}} = \epsilon^{p_1 t}.$$

Hence,

$$i = \frac{E}{L(p_2 - p_1)}\{\epsilon^{p_2 t} - \epsilon^{p_1 t}\}. \tag{19}$$

Substituting the values of p_2 and p_1 from (17), we get

$$\begin{aligned} i &= \frac{E\epsilon^{-\alpha t}}{-2Lj\beta}\{\epsilon^{-j\beta t} - \epsilon^{j\beta t}\} \\ &= \frac{E\epsilon^{-\alpha t} \sin \beta t}{L\beta}. \end{aligned} \tag{20}$$

In the problems considered above, the application of a steady voltage was assumed. The operational method, however, is not at all restricted to that particular type of voltage; it is equally effective in the solution of problems for alternating voltages. Since, however, a periodic alternating voltage of any form may be resolved into simple harmonic components, it will be sufficient to consider only the case of applied simple-harmonic voltage.

Inductive Circuit, Applied Alternating Voltage.—In the case of an inductive circuit, applied voltage simple harmonic, $E \cos \omega t$, the circuit equation is

$$(Lp + R)i = E \cos \omega t,$$

and

$$i = \frac{E \cos \omega t}{Lp + R};$$

or

$$i = \frac{E(\epsilon^{j\omega t} + \epsilon^{-j\omega t})}{2Lp\left(1 + \frac{R}{Lp}\right)}. \tag{21}$$

To algebrize the above equation involves operating on an exponential function by the operator $\left(1 + \frac{R}{LP}\right)^{-1}$. It is desirable to establish first a general formula for this operation. Consider the following equation:

$$u = \frac{1}{1 + \frac{a}{p}}\epsilon^{\lambda t}. \tag{22}$$

By division,

$$u = \left\{1 - \frac{a}{p} + \frac{a^2}{p^2} - \frac{a^3}{p^3} + \cdots\right\}\epsilon^{\lambda t}, \tag{23}$$

which implies successive integrations of the exponential term from 0 to t. Thus:

$$p^{-1}\epsilon^{\lambda t} = \int_0^t \epsilon^{\lambda t}dt = \frac{1}{\lambda}(\epsilon^{\lambda t} - 1)$$

$$p^{-2}\epsilon^{\lambda t} = \int_0^t \frac{1}{\lambda}(\epsilon^{\lambda t} - 1)dt = \frac{1}{\lambda^2}(\epsilon^{\lambda t} - 1) - \frac{t}{\lambda}$$

$$p^{-3}\epsilon^{\lambda t} = \int_0^t \left\{\frac{1}{\lambda^2}(\epsilon^{\lambda t} - 1) - \frac{t}{\lambda}\right\}dt = \frac{1}{\lambda^3}(\epsilon^{\lambda} - 1) - \frac{t}{\lambda^2} - \frac{t^2}{\lambda 2!}.$$

Repeating the process over and over, we get for the nth integration

$$p^{-n}\epsilon^{\lambda t} = \frac{1}{\lambda^n}(\epsilon^{\lambda t} - 1) - \frac{t}{\lambda^{n-1}} - \frac{t^2}{\lambda^{n-2}2!} - \frac{t^3}{\lambda^{n-3}3!} \cdots$$

Substituting these integration values in (23) and combining terms of the same powers of t, we obtain the following:

$$\begin{aligned} u = &\left(1 - \frac{a}{\lambda} + \frac{a^2}{\lambda^2} - \frac{a^3}{\lambda^3} + \cdots\right)\epsilon^{\lambda t} \\ &+ \frac{a}{\lambda}\left(1 - \frac{a}{\lambda} + \frac{a^2}{\lambda^2} - \frac{a^3}{\lambda^3} + \cdots\right) \\ &- \frac{a}{\lambda}(at)\left(1 - \frac{a}{\lambda} + \frac{a^2}{\lambda^2} - \frac{a^3}{\lambda^3} + \cdots\right) \\ &+ \frac{a}{\lambda}\frac{(at)^2}{2!}\left(1 - \frac{a}{\lambda} + \frac{a^2}{\lambda^2} - \frac{a^3}{\lambda^3} + \cdots\right) \\ &+ \cdots \end{aligned}$$

Each of the bracket terms is the expansion of the fraction $\dfrac{1}{1+\dfrac{a}{\lambda}}$, hence,

$$u = \frac{1}{1+\dfrac{a}{\lambda}}\left\{\epsilon^{\lambda t} + \frac{a}{\lambda}\left(1 - at + \frac{(at)^2}{2!} - \frac{(at)^3}{3!} + \cdots\right)\right\}$$

$$= \frac{\epsilon^{\lambda t} + \dfrac{a}{\lambda}\epsilon^{-at}}{1+\dfrac{a}{\lambda}} = \frac{\epsilon^{\lambda t}}{1+\dfrac{a}{\lambda}} + \frac{\epsilon^{-at}}{1+\dfrac{\lambda}{a}}. \tag{24}$$

By the aid of this formula, (21) can be algebrized at sight, replacing λ by $j\omega$ or $-j\omega$, and a by R/L. This gives the following result:

$$i = \frac{E}{2Lp}\left\{\frac{\epsilon^{j\omega t}}{1+\dfrac{R}{Lj\omega}} + \frac{\epsilon^{-j\omega t}}{1-\dfrac{R}{Lj\omega}} + \frac{\epsilon^{-\frac{R}{L}t}}{1+j\dfrac{L\omega}{R}} + \frac{\epsilon^{-\frac{R}{L}t}}{1-j\dfrac{L\omega}{R}}\right\}. \tag{25}$$

Operating on the bracket terms of (25) by $1/p$, that is, integrating between the limits 0 and t, we obtain:

$$i = \frac{E}{2}\left\{\frac{\epsilon^{j\omega t}}{R+Lj\omega} - \frac{1}{R+Lj\omega} + \frac{\epsilon^{-j\omega t}}{R-Lj\omega} - \frac{1}{R-Lj\omega}\right.$$
$$\left. - \frac{\epsilon^{-\frac{R}{L}t}}{R+jL\omega} + \frac{1}{R+Lj\omega} - \frac{\epsilon^{-\frac{R}{L}t}}{R-jL\omega} + \frac{1}{R-jL\omega}\right\}$$

$$= \frac{E}{2}\left\{\frac{\epsilon^{j\omega t}}{R+Lj\omega} + \frac{\epsilon^{-j\omega t}}{R-Lj\omega} - \frac{2R\epsilon^{-\frac{R}{L}t}}{R^2+L^2\omega^2}\right\}. \tag{26}$$

Replacing $\epsilon^{j\omega t}$ by $\cos\omega t + j\sin\omega t$, and $\epsilon^{-j\omega t}$ by $\cos\omega t - j\sin\omega t$ and combining, the above simplifies to the following:

$$i = E\left\{\frac{R\cos\omega t + L\omega\sin\omega t}{R^2+L^2\omega^2} - \frac{R\epsilon^{-\frac{R}{L}t}}{R^2+L^2\omega^2}\right\}$$

$$= \frac{E\cos(\omega t - \varphi)}{\sqrt{R^2+L^2\omega^2}} - \frac{ER\epsilon^{-\frac{R}{L}t}}{R^2+L^2\omega^2}. \tag{27}$$

$$\tan\varphi = \frac{L\omega}{R}.$$

The first right-hand term of (26) is the steady component current, and the second term the transient component, which decreases to zero as t increases.

The derivation of the above expression was carried through to how the straightforward process in operational solutions. The same result can be arrived at by a simpler method, utilizing the "shift principle"; that is, shifting the operand, the exponential factor, to the left, suitably modifying the operator, and operating on unity function. This will be clear from the following mathematical considerations:

Shifting of Operand.—If $X(t)$ denotes an algebraical rational function of t which can be expanded in ascending or descending powers (or both) of the variable, the following relations obtain:

$$f(p)\epsilon^{at} = f(a)\epsilon^{at} \tag{28}$$

For since p stands for d/dt, we have

$$p\epsilon^{at} = a\epsilon^{at}$$

Operating on both sides by p^{-1}, we get

$$p^{-1}p\epsilon^{at} = p^{-1}a\epsilon^{at}$$

But $p^{-1}p = 1$, hence,

$$a^{-1}\epsilon^{at} = p^{-1}\epsilon^{at}$$

Repeating the operation, we get

$$pp\epsilon^{at} = pa\epsilon^{at} = a^2\epsilon^{at},$$

also,

$$p^{-1}p^{-1}\epsilon^{at} = p^{-1}a^{-1}\epsilon^{at} = a^{-2}\epsilon^{at}.$$

By repeated operations, we obtain the following relations:

$$\left.\begin{aligned} p^m\epsilon^{at} &= a^m\epsilon^{at}, \\ p^{-m}\epsilon^{at} &= a^{-m}\epsilon^{at}. \end{aligned}\right\} \tag{29}$$

Now as $f(p)$ is an algebraical function which can be expanded in powers, we may write,

$$\begin{aligned} f(p)\epsilon^{at} &= \{A_0 + A_1p + A_2p^2 + A_3p^3 + \cdots \\ &\qquad B_1p^{-1} + B_2p^{-2} + B_3p^{-3} + \cdots\}\epsilon^{at} \\ &= \{A_0 + A_1a + A_2a^2 + A_3a^3 + \cdots \\ &\qquad B_1a^{-1} + B_2a^{-2} + B_3a^{-3} + \cdots\}\epsilon^{at} \\ &= f(a)\epsilon^{at}. \end{aligned} \tag{30}$$

which establishes the relation (27).

If T denotes any function whatever of t, then

$$f(p)\{\epsilon^{at}T\} = \epsilon^{at}f(p + a)T. \tag{31}$$

that is, changing p to $p + a$, shifts the factor ϵ^{at} to the left, and the operand is reduced to T. The proof of this is as follows: A single operation with p gives

$$p\epsilon^{at}T = a\epsilon^{at}T + \epsilon^{at}pT = \epsilon^{at}(p + a)T.$$

Repeat the operation,

$$\begin{aligned} pp\epsilon^{at}T &= p\epsilon^{at}(p + a)T \\ &= a\epsilon^{at}(p + a)T + \epsilon^{at}p(p + a)T \\ &= \epsilon^{at}(p + a)(p + a)T. \end{aligned}$$

Hence,

$$p^2\epsilon^{at}T = \epsilon^{at}(p + a)^2T.$$

By repeated operations, we get the result

Obviously,

$$\left.\begin{aligned} p^n\epsilon^{at}T &= \epsilon^{at}(p + a)^nT. \\ p^n\epsilon^{-at}T &= \epsilon^{-at}(p - a)^nT. \end{aligned}\right\} \tag{32}$$

Consider now the case of negative indices; write,

$$(p + a)^nT = T_1;\ T = (p + a)^{-n}T_1,$$

introduce this value of T in the first equation (32),

$$p^n\epsilon^{at}(p + a)^{-n}T_1 = \epsilon^{at}T_1.$$

Operate on each side by p^{-n}, and the result is

$$\epsilon^{at}(p + a)^{-n}T_1 = p^{-n}\epsilon^{at}T_1.$$

Now, no limitations were assigned to the form of T, and there is, therefore, none to T_1, which can represent any function of T; replacing it, therefore, by T, we get

Similarly,

$$\left.\begin{aligned} p^{-n}\epsilon^{at}T &= \epsilon^{at}(p + a)^{-n}T. \\ p^{-n}\epsilon^{-at}T &= \epsilon^{-at}(p - a)^{-n}T. \end{aligned}\right\} \tag{33}$$

Since $f(p)$ is expressible as a series in ascending and descending powers of p, we can for each term separately shift the factor ϵ^{at} to the left and change p to $p + a$ in each term. There results the relation

$$f(p)\epsilon^{at}T = \epsilon^{at}f(p + a)T. \tag{34}$$

That is, in operating on a time function which has a factor ϵ^{at}, this factor may be shifted to the left by changing every p in the operator to $p + a$. If the entire operand is ϵ^{at}, then shifting it to the left and changing p to $p + a$ changes the operand to unity time function.

Solution of Inductive Circuit Problem Utilizing the Shift Principle.—The utilization of the shift principle established in the preceding section is frequently very useful in operational solutions. As an illustration of the application of the shift principle, we may consider the same problem, that of the cur-

rent rise in an inductive circuit under the application of an alternating voltage. By (21), we have

$$i = \frac{E}{2}\left\{\frac{\epsilon^{j\omega t}}{Lp + R} + \frac{\epsilon^{-j\omega t}}{Lp + R}\right\} \tag{35}$$

Consider each right-hand term of (34) separately. For the first term, shift the exponential factor to the left, and change p to $p + j\omega$ in accordance with (34); we get

$$\frac{\epsilon^{j\omega t}}{Lp + R} = \epsilon^{j\omega t}\frac{1}{L(p + j\omega) + R} = \frac{\epsilon^{j\omega t}}{Lp}\left(1 + \frac{R + Lj\omega}{Lp}\right)^{-1}, \tag{36}$$

and, in this case, the operand is unity. Operating by the bracket term on unity, we have, by (12),

$$\left(1 + \frac{R + Lj\omega}{Lp}\right)^{-1} = \epsilon^{-\frac{(R + Lj\omega)t}{L}},$$

and this, again, is to be operated on by $1/p$; that is, integrating from 0 to t, which gives

$$\int_0^t \epsilon^{-\frac{(R+Lj\omega)t}{L}}\,dt = -\frac{L}{R + Lj\omega}\left\{\epsilon^{-\frac{R+Lj\omega t}{L}} - 1\right\}.$$

Introducing this in (36) gives

$$\frac{\epsilon^{j\omega t}}{Lp + R} = -\frac{\epsilon^{j\omega t}}{R + Lj\omega}\left\{\epsilon^{-\frac{R+Lj\omega t}{L}} - 1\right\}$$

$$= -\frac{\epsilon^{-\frac{R}{L}t}}{R + Lj\omega} + \frac{\epsilon^{j\omega t}}{R + Lj\omega}. \tag{37}$$

Carrying through the same operation in connection with the second right-hand term of (35), the same result will follow except for the change in sign before $j\omega$; that is,

$$\frac{\epsilon^{-j\omega t}}{Lp + R} = -\frac{\epsilon^{-\frac{R}{L}t}}{R - Lj\omega} + \frac{\epsilon^{-j\omega t}}{R - Lj\omega}. \tag{38}$$

Introducing the values from (37) and (38) in (35) gives the expression for the current.

$$i = \frac{E}{2}\left\{\frac{\epsilon^{j\omega t}}{R + Lj\omega} + \frac{\epsilon^{-j\omega t}}{R - Lj\omega} - \frac{2R}{R^2 + L^2\omega^2}\epsilon^{-\frac{R}{L}t}\right\}, \tag{39}$$

which is the same as (25).

It is seen that, in this problem, at least, utilizing the shift principle has introduced a considerable simplification in the solution. It is a useful principle, which can, in some cases, be taken advantage of to facilitate operational solutions.

Circuit of Capacity and Resistance Applied Alternating Voltage.—The circuit equation is as follows:

$$i\left(R + \frac{1}{Cp}\right) = E \cos \omega t,$$

$$i = \frac{E \cos \omega t}{R + \dfrac{1}{Cp}} = \frac{ECp(\epsilon^{j\omega t} + \epsilon^{-j\omega t})}{2(1 + RCp)}. \qquad (40)$$

Shifting the operands $\epsilon^{j\omega t}$ and $\epsilon^{-j\omega t}$ to the left, we get

$$i = \frac{E\epsilon^{j\omega t}}{2}\left\{\frac{C(p + j\omega)}{1 + RC(p + j\omega)}\right\} + \frac{E\epsilon^{-j\omega t}}{2}\left\{\frac{C(p - j\omega)}{1 + RC(p - j\omega)}\right\}. \quad (41)$$

Each of the bracket terms is to operate on unity time function. Consider each bracket term separately.

$$\frac{C(p + j\omega)}{1 + RC(p + j\omega)} = \frac{\dfrac{1}{R} + \dfrac{j\omega}{Rp}}{1 + \dfrac{1 + RCj\omega}{RCp}}$$

operating on unity by $\dfrac{1}{1 + \dfrac{1 + RCj\omega}{RCp}}$ gives, by (12), $\epsilon^{-\left(\frac{1}{RC} + j\omega\right)t}$,

and this, again, is to be operated on by $\left(\dfrac{1}{R} + \dfrac{j\omega}{Rp}\right)$.

Hence,

$$\frac{C(p + j\omega)}{1 + RC(p + j\omega)} = \frac{1}{R}\epsilon^{-\left(\frac{1}{RC}+j\omega\right)t} + j\frac{\omega}{R}\int_0^t \epsilon^{-\left(\frac{1}{RC}+j\omega\right)t}\,dt$$

$$= \frac{1}{R}\epsilon^{-\left(\frac{1}{RC}+j\omega\right)t} - \frac{1}{R + \dfrac{1}{Cj\omega}}\epsilon^{-\left(\frac{1}{RC}+j\omega\right)t} + \frac{1}{R + \dfrac{1}{Cj\omega}}. \qquad (41)$$

In a similar way, we get for the second right-hand bracket term

$$\frac{C(p - j\omega)}{1 + RC(p - j\omega)} = \frac{1}{R}\epsilon^{-\left(\frac{1}{RC}-j\omega\right)t} - \frac{1}{R - \dfrac{1}{Cj\omega}}\epsilon^{-\left(\frac{1}{RC}-j\omega\right)t} +$$

$$\frac{1}{R - \dfrac{1}{Cj\omega}}. \qquad (42)$$

Introducing the values from (41) and (42) into (40) gives

$$i = \frac{E}{2}\left\{\frac{1}{R}\epsilon^{-\frac{t}{RC}} - \frac{1}{R + \frac{1}{Cj\omega}}\epsilon^{-\frac{t}{RC}} + \frac{1}{R + \frac{1}{Cj\omega}}\epsilon^{j\omega t} + \frac{1}{R}\epsilon^{-\frac{1}{RC}t} - \frac{1}{R - \frac{1}{Cj\omega}}\epsilon^{-\frac{t}{RC}} + \frac{1}{R - \frac{1}{Cj\omega}}\epsilon^{j\omega t}\right\}. \tag{43}$$

This transforms by simple algebra into the following:

$$i = E\left\{\frac{\epsilon^{-\frac{t}{RC}}}{R(1 + R^2C^2\omega^2)} + \frac{\left(R\cos\omega t - \frac{1}{C\omega}\sin\omega t\right)}{R^2 + \frac{1}{C^2\omega^2}}\right\},$$

$$= E\left\{\frac{\epsilon^{-\frac{t}{RC}}}{R(1 + R^2C^2\omega^2)} + \frac{\cos(\omega t + \psi)}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}}\right\}, \tag{44}$$

$$\tan\psi = \frac{1}{RC\omega}.$$

The first term is the transient component, and the second term the permanent-current component. For $\omega = 0$, that is, steady voltage, (44) reduces to

$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}.$$

The derivation of the solutions to the above problems by the operational methods illustrates the general process of operational solutions. Simple problems were selected with a view to emphasizing the process rather than showing the application to the solution of more difficult problems. These will also serve as an introduction to the discussion of the expansion theorem, taken up in the next chapter, which is a development of the operational methods but offers a more direct way of obtaining a solution, avoiding many intermediate steps and saving much work. Other illustrations of direct operational methods will be given in the discussion of circuits of distributed electrical constants.

CHAPTER II

EXPANSION THEOREM

The importance of the expansion theorem in the solution of problems in electric-circuit theory cannot be overemphasized. It applies to all kinds of circuits; concentrated inductances and capacities, or distributed inductances and capacities, giving directly the solution for the current and voltage distribution in the circuits; the steady as well as the transient state. Of course, its greatest utility is in the facility it affords for obtaining solutions for transient effects, which may be, in some cases, very difficult if not altogether impossible to obtain by the usual methods.

Why this tremendously useful theorem should have remained so long unappreciated and not made more extensive use of can be explained, perhaps, as due to the fact that Heaviside's mathematical methods, because of their originality and novelty, were not fully comprehended for some time and were, therefore, neglected. Within recent years, however, more attention has been given by physicists and engineers to Heaviside's work, and quite a number of papers were written to elucidate and amplify his writings.

The mathematical formulation of the expansion theorem is as follows:

$$i = \frac{E}{Z(p)_{p=0}} + \sum_{p_n} \frac{\epsilon^{p_n t}}{\frac{\partial Z(p)}{\partial p}_{p = p_n}}. \tag{1}$$

The meaning of it is this: Given any network of circuits, and a steady voltage applied at any point in the system, the current is given by the above expression; the first right-hand term represents the steady-state component, and the summation term gives the transient component. $Z(p)$ is the generalized impedance of the system. $Z(p)_{p=0}$ is the value of $Z(p)$ when $p = 0$, and the summation term is to extend to all the roots of the equation $Z(p) = 0$. The meaning and method of using it will be clearer as we go through with the process of derivation and the application to the solution of various problems.

The method and reasoning by which Heaviside arrived at this theorem is rather difficult to follow, involving, as it does, closely theoretic reasoning of a dynamical system, this is given in Chap. VIII. It is not at all necessary, however, that we follow his method; the derivation may be obtained by a simpler method involving only algebraic processes, which the author has given in a paper published in the *Journal of the Franklin Institute,* December, 1922.

DERIVATION OF EXPANSION FORMULA

Given any network of circuits, the combined impedance of the circuit system being designated by $Z(p)$, then, obviously, the current is given by

$$i = \frac{E}{Z(p)}. \tag{2}$$

$Z(p)$ always contains p in various powers, and, in general, $Z(p)$ can be written,

$$Z(p) = p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots, \tag{3}$$

Let the roots corresponding to $Z(p) = 0$ be p_1, p_2, p_3, etc., then (3) may be written in this form:

$$Z(p) = (p - p_1)(p - p_2)(p - p_3) \cdots,$$

and

$$i = \frac{E}{(p - p_1)(p - p_2)(p - p_3) \cdots}. \tag{4}$$

The denominator of (4) may be resolved by partial fractions into the following:

$$\frac{1}{(p - p_1)(p - p_2)(p - p_3) \cdots} = \frac{A}{p - p_1} + \frac{B}{p - p_2} + \frac{C}{p - p_3} + \cdots \frac{M}{p - p_m}, \tag{5}$$

where

$$\left.\begin{aligned} A &= \frac{1}{(p_1 - p_2)(p_1 - p_3)(p_1 - p_4) \cdots (p_1 - p_m)}, \\ B &= \frac{1}{(p_2 - p_1)(p_2 - p_3)(p_2 - p_4) \cdots (p_2 - p_m)}, \\ &\cdots\cdots\cdots\cdots\cdots \\ M &= \frac{1}{(p_m - p_1)(p_m - p_2)(p_m - p_3) \cdots (p_m - p_{m-1})}. \end{aligned}\right\} \tag{6}$$

For the explanation of the above transformation, see any algebra. The following notation is convenient:

$$A = \frac{1}{Z(p_1)};\ B = \frac{1}{Z(p_2)};\ C = \frac{1}{Z(p_3)}\ \cdot\ \cdot\ \cdot$$

Introducing these in (2), we get

$$i = E\left\{\frac{1}{(p - p_1)Z(p_1)} + \frac{1}{(p - p_2)Z(p_2)} + \cdot\ \cdot\ \cdot \right.$$
$$\left. \frac{1}{(p - p_m)Z(p_m)}\right\}. \qquad (7)$$

Now write each of the factors $\frac{1}{p - p_m}$ in the form $\frac{1}{p\left(1 - \frac{p_m}{p}\right)} = \frac{1}{p}\left(1 - \frac{p_m}{p}\right)^{-1}$, and expanding in a series, we obtain

$$\frac{1}{p - p_m} = \frac{1}{p}\left\{1 + \frac{p_m}{p} + \frac{p_m^2}{p^2} + \frac{p_m^3}{p^3} + \cdot\ \cdot\ \cdot\right\}$$
$$= \frac{1}{p}\left\{1 + p_m t + \frac{p_m^2 t^2}{2!} + \frac{p_m^3 t^3}{3!} + \cdot\ \cdot\ \cdot\right\}$$
$$= \frac{1}{p}\,\epsilon^{p_m t}$$

Operating on $\epsilon^{p_m t}$ by $1/p$, which means integration to t from 0 to t, we get

$$\frac{1}{p - p_m} = \int_0^t \epsilon^{p_m t}dt = \frac{1}{p_m}\,(\epsilon^{p_m t} - 1) \qquad (8)$$

Substituting the relation given by (8) for the factors in each of the terms of (7), we get the following:

$$i = E\left\{\frac{\epsilon^{p_1 t}}{p_1 Z(p_1)} + \frac{\epsilon^{p_2 t}}{p_2 Z(p_2)} + \cdot\ \cdot\ \cdot + \frac{\epsilon^{p_m t}}{p_m Z(p_m)} - \right.$$
$$\left. \frac{1}{p_1 Z(p_1)} - \frac{1}{p_2 Z(p_2)} - \cdot\ \cdot\ \cdot\right\} \qquad (9)$$

Now suppose we write,

$$Z(p) = (p - p_1)F(p)$$

where

$$F(p) = (p - p_2)(p - p_3)(p - p_4)\ \cdot\ \cdot\ \cdot$$

By differentiation, we obtain

$$\frac{\partial Z(p)}{\partial p} = (p - p_1)\,\frac{\partial F(p)}{\partial p} + F(p).$$

For

$$p = p_1;\ p - p_1 = 0;\ \text{and}\ F(p) = Z(p_1).$$

Hence,

$$\left.\begin{aligned} &\frac{\partial Z(p)}{\partial p}\,p = p_1 = Z(p_1). \\ &\text{Similarly,} \\ &\frac{\partial Z(p)}{\partial p}\,p = p_2 = Z(p_2), \end{aligned}\right\} \qquad (10)$$

and so on for all other values of p.
Also, for $p = 0$ by (5) and (6),

$$\frac{1}{Z(p)_{p=0}} = -\frac{1}{p_1 Z(p_1)} - \frac{1}{p_2 Z(p_2)} - \frac{1}{p_3 Z(p_3)} - \cdots \qquad (11)$$

Substituting the results from (10) and (11) in (9), we finally obtain

$$i = \frac{E}{Z(p)_{p=0}} + E\left\{ \frac{\epsilon^{p_1 t}}{p_1 \dfrac{\partial Z(p)}{\partial p}\,p = p_1} + \frac{\epsilon^{p_2 t}}{p_2 \dfrac{\partial Z(p)}{\partial p}\,p = p_2} + \cdots \right\} \qquad (12)$$

We may write the above in a more convenient form,

$$i = \frac{E}{Z(p)_{p=0}} + E\sum_{n=1}^{n=m} \frac{\epsilon^{p_n t}}{p_n \dfrac{\partial Z(p)}{\partial p}\,p = p_n} \qquad (13)$$

which is the form given by equation (1).

The physical interpretation of the formula is this: Given any electrical circuit and a steady electromotive force impressed on it, the current in the system is the algebraic sum of two components, one of which is the steady-current component, the first term of the formula in which $p = 0$; that is, the reactance terms are zero and only the resistances are effective; and the summation term represents the transient current, the number of terms occurring in the summation term depending on the number of degrees of freedom of the system, the number of roots of the equation $Z(p) = 0$. Also, the transient terms may be in the form to show simple decaying currents or decaying oscillatory currents, depending on the character of the circuit system. Further, the summation term represents, also, the subsidence

of the current in a circuit system when the electromotive force is suddenly removed.

It is to be observed that, in the derivation of the expansion-theorem formula, it was assumed that all the roots of the equation $Z(p) = 0$ are unequal. For the case where some of the roots are equal, the problem is more complex, and we can refer here only to an interesting discussion of it by Goto.[1] Formula (13), however, is quite sufficient for most practical cases.

Extension of Expansion Theorem for Alternating Electromotive Forces.—Any alternating e.m.f. may be resolved into sinusoidal components, and it is, therefore, sufficient to consider only the case of asinusoidal e.m.f. Assume the e.m.f. impressed on the circuit system to be represented by the real part of $E\epsilon^{j\omega t}$, and put for brevity $\lambda = j\omega$, equation (2) assumes the form

$$i = \frac{E}{Z(p)}\epsilon^{\lambda t} \tag{14}$$

Resolving $Z(p)$ by partial fractions, as in the preceding case, we get, as before,

$$i = E\left\{\frac{1}{(p-p_1)Z(p_1)} + \frac{1}{(p-p_2)Z(p_2)} + \cdots + \frac{1}{(p-p_m)Z(p_m)}\right\}. \tag{15}$$

Operating by $1/p$ on $\epsilon^{\lambda t}$ gives

$$\frac{1}{p}\epsilon^{\lambda t} = \int_0^t \epsilon^{\lambda t}dt = \frac{1}{\lambda}(\epsilon^{\lambda t} - 1),$$

$$\frac{1}{p^2}\epsilon^{\lambda t} = \int_0^t \frac{1}{\lambda}(\epsilon^{\lambda t} - 1)dt = \frac{1}{\lambda^2}(\epsilon^{\lambda t} - 1) - \frac{t}{\lambda}.$$

Repeating the process over and over, we get

$$\frac{1}{p^m}\epsilon^{\lambda t} = \frac{1}{\lambda^m}\epsilon^{\lambda t} - \frac{1}{\lambda^m} - \frac{1}{\lambda^{m-1}}t - \frac{1}{\lambda^{m-2}}\frac{t^2}{2!} - \frac{1}{\lambda^{m-3}}\frac{t^3}{3!} - \cdots \tag{16}$$

Now expanding the term $\frac{1}{p - p_1}$ as previously and operating on $\epsilon^{\lambda t}$, using (16), we obtain

$$\frac{1}{p-p_1}\epsilon^{\lambda t} = \frac{1}{p}\left\{\epsilon^{\lambda t}\left(1 + \frac{p_1}{\lambda} + \frac{p_1^2}{\lambda^2} + \frac{p_1^3}{\lambda^3} + \cdots\right) - \frac{p_1}{\lambda}\left(1 + \frac{p_1}{\lambda} + \frac{p_1^2}{\lambda_2} + \cdots\right)\right.$$

[1] Goto Machinori, "Extension of the Heaviside Expansion Theorem," Researches of the Electrotechnical Laboratory, Tokyo, Japan.

$$-\frac{p_1^2}{\lambda}t\left(1+\frac{p_1}{\lambda}+\frac{p_1^2}{\lambda^2}+\frac{p_1^3}{\lambda^3}+\cdots\right)-\frac{p_1^3}{\lambda}\frac{t^2}{2!}\left(1+\frac{p_1}{\lambda}+\frac{p_1^2}{\lambda^2}+\frac{p_1^3}{\lambda^3}+\cdots\right)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \Big\}$$

But

$$1+\frac{p_1}{\lambda}+\frac{p_1^2}{\lambda^2}+\frac{p_1^3}{\lambda^3}+\cdots \frac{1}{1-\dfrac{p_1}{\lambda}},$$

hence,

$$\frac{1}{p-p_1}\epsilon^{\lambda t}=\frac{1}{1-\dfrac{p_1}{\lambda}}\frac{1}{p}\left\{\epsilon^{\lambda t}-\frac{p_1}{\lambda}\left(1+p_1t+\frac{p_1^2t^2}{2!}+\frac{p_1^3t^3}{3!}+\cdots\right)\right\}$$

$$=\frac{1}{1-\dfrac{p_1}{\lambda}}\frac{1}{p}\left\{\epsilon^{\lambda}-\frac{p_1}{\lambda}\epsilon^{p_1t}\right\}.$$

Operating now on the bracket term by $1/p$, we finally obtain

$$\frac{1}{p-p_1}\epsilon^{\lambda t}=\frac{1}{\lambda-p_1}\{\epsilon^{\lambda t}-\epsilon^{p_1t}\}. \tag{17}$$

In a similar way, for the other terms of (15),

$$\frac{1}{p-p_2}=\frac{1}{\lambda-p_2}\{\epsilon^{\lambda t}-\epsilon^{p_2t}\},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{1}{p-p_m}=\frac{1}{\lambda-p_m}\{\epsilon^{\lambda t}-\epsilon^{p_mt}\}.$$

Substituting these values, equation (15) transforms to

$$i=E\,\epsilon^{\lambda t}\left\{\frac{1}{(\lambda-p_1)Z(p_1)}+\frac{1}{(\lambda-p_2)Z(p_2)}+\frac{1}{(\lambda-p_3)Z(p_3)}+\cdots\right\}$$

$$-E\left\{\frac{\epsilon^{p_1t}}{(\lambda-p_1)Z(p_1)}+\frac{\epsilon^{p_2t}}{(\lambda-p_2)Z(p_2)}+\frac{\epsilon^{p_3t}}{(\lambda-p_3)Z(p_3)}+\cdots\right\} \tag{18}$$

The first bracket term is $\dfrac{1}{Z(p)_{p=\lambda}}$, which is obvious on comparing with (5) and (6). In the second bracket term, $Z(p_n)$ may be replaced by $\dfrac{\partial Z(p)}{\partial p}\ p = p_n$ by equation (10). Hence, (18) reduces to the following:

$$i = \frac{E\epsilon^{\lambda t}}{Z(p)_{p=\lambda}} - E\sum_{n=1}^{n=m} \frac{\epsilon^{p_n t}}{(\lambda - p_n)\dfrac{\partial Z(p)}{\partial p}\ p = p_n}. \tag{19}$$

Replacing λ by $j\omega$, we have, finally,

$$i = \frac{E\epsilon^{j\omega t}}{Z(p)_{p=j\omega}} - E\sum_{n=1}^{n=m} \frac{\epsilon^{p_n t}}{(j\omega - p_n)\dfrac{\partial Z(p)}{\partial p}\ p = p_n}. \tag{20}$$

This we may consider the modified expansion formula applicable for alternating voltage, and which we shall refer to hereafter as the *second expansion formula.* The first right-hand term of (20) is the permanent-current component, and the summation term the transient component. It is clear, of course, that only the real parts are to be used in the solution of any problem by the application of this formula. In the summation term, however, the imaginary part is readily eliminated. We may write (20) in this form:

$$i = \frac{E\epsilon^{j\omega t}}{Z(p)_{p=j\omega}} + E\sum_{n=1}^{n=m} \frac{p_n\epsilon^{p_n t}}{(p_n + \omega^2)\dfrac{\partial Z(p)}{\partial p}\ p = p_n}. \tag{21}$$

Thus, the steady-current component is expressed in complex quantities of which the real part only is to be used in the final solution, and the summation term giving the real solution for the transient components.

If the applied voltage is at some phase angle θ, that is $E\cos(\omega t + \theta)$, then in place of $E\epsilon^{j\omega t}$, we have $E\epsilon^{j(\omega t+\theta)} = E\epsilon^{j\theta}\epsilon^{j\omega t}$, and in the final formula, E is to be multiplied by the factor $\epsilon^{j\theta}$, which introduces a modification in (20) as follows:

$$i = \frac{E\epsilon^{j(\omega t+\theta)}}{Z(p)_{p=j\omega}} - E\epsilon^{j\theta}\sum_{n=1}^{n=m} \frac{\epsilon^{p_n t}}{(j\omega - p_n)\dfrac{\partial Z(p)}{\partial p}\ p = p_n}. \tag{22}$$

Taking only the real part of the summation term gives

$$i = \frac{E\epsilon^{j(\omega t+\theta)}}{Z(p)_{p=j\omega}} + E\sum_{n=1}^{n=m} \frac{\cos(\varphi_n + \theta)\epsilon^{p_n t}}{\sqrt{p_n^2 + \omega^2}\frac{\partial Z(p)}{\varphi p}\, p = p_n}. \qquad (23)$$

$$\tan \varphi_n = \frac{\omega}{p_n}.$$

We shall now consider a number of problems to illustrate the application of the expansion formula to the solution of electric-circuit problems. We shall confine the discussion in this chapter to problems relating to circuits of concentrated inductance and capacity. Other illustrations of the application of the expansion formulae to the solution of more difficult problems, those relating to circuits of distributed electrical constants, are given in the chapters following.

Circuit of Inductance L and Resistance R.—Assume a steady voltage applied to an inductive circuit, the current equation is

$$L\frac{di}{dt} + Ri = E,$$

and

$$i = \frac{E}{Lp + R}. \qquad (24)$$

The determinantal equation for this case is

$$Z(p) = Lp + R = 0$$

a one-degree equation, only one root, $p = -R/L$;

$$\frac{\partial Z(p)}{\partial p} = L; \text{ and, for } p = 0,\ Z(p)_{p=0} = R.$$

Substituting these values in (13) gives the solution of the problem,

$$i = \frac{E}{R} - \frac{E}{\frac{R}{L}L}\epsilon^{-\frac{R}{L}t},$$

$$= \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right), \qquad (25)$$

the well-known formula for the current rise in an inductive circuit, obtained in the preceding chapter by direct operational process.

For alternating e.m.f., apply the second expansion formula (21), which gives

$$i = \frac{E\epsilon^{j\omega t}}{Lj\omega + R} - \frac{E\frac{R}{L}\epsilon^{-\frac{R}{L}t}}{\left(\frac{R^2}{L^2} + \omega^2\right)L},$$

$$= \frac{E\epsilon^{j\omega t}}{Lj\omega + R} - \frac{ER}{R^2 + L^2\omega^2}\epsilon^{-\frac{R}{L}t}. \tag{26}$$

Taking only the real part of the first right-hand term,

$$i = \frac{E}{R^2 + L^2\omega^2}\left\{R\cos\omega t + L\omega\sin\omega t - R\epsilon^{-\frac{R}{L}t}\right\},$$

$$= \frac{E}{\sqrt{R^2 + L^2\omega^2}}\left\{\cos(\omega t - \varphi) - \cos\varphi\epsilon^{-\frac{R}{L}t}\right\}. \tag{27}$$

$$\tan\varphi = \frac{L\omega}{R}.$$

The first term in the above equation is the permanent alternating-current component, and the second term the transient component. For $t = 0$, $i = 0$ as we should expect, and for $t = \infty$

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\cos(\omega t - \varphi),$$

only the permanent-current component active.

Circuit of Capacity C and Resistance R.—The voltage across a condenser for a variable current in the circuit is $V = 1/C\int idt$, and the circuit equation, therefore, is

$$\left(R + \frac{1}{Cp}\right)i = E,$$

and

$$i = \frac{E}{R + \frac{1}{Cp}}. \tag{28}$$

The determinantal equation is

$$Z(p) = R + \frac{1}{Cp} = 0$$

which gives only one value of p.

$$p = -\frac{1}{RC}$$

$$\frac{\partial Z(p)}{\partial p} = -\frac{1}{Cp^2} = -R^2C$$

For steady voltage,

$$Z(p)_{p=0} = \infty \text{ because } \frac{1}{Cp} = \infty \text{ for } p = 0.$$

The steady-current component is zero, in this case, as we should expect. Substituting the values of p and $\partial Z(p)/\partial p$ in the summation term of (13), we obtain the following expression for the current in the circuit:

$$i = \frac{E\epsilon^{-\frac{1}{RC}t}}{\frac{1}{RC}R^2C} = \frac{E}{R}\epsilon^{-\frac{1}{RC}t}. \tag{29}$$

the well-known expression for the charging current of a condenser circuit.

For an applied alternating voltage, $E\epsilon^{j\omega t}$, use the same values of p and $\partial Z(p)/\partial p$ in formula (21).

$$i = \frac{E\epsilon^{j\omega t}}{R + \frac{1}{Cj\omega}} + \frac{E\frac{1}{RC}\epsilon^{-\frac{1}{RC}t}}{\left(\frac{1}{R^2C^2} + \omega^2\right)R^2C}$$

Taking only the real part of the first right-hand term, and rearranging slightly the second term, gives the following:

$$i = \frac{E\left(R\cos\omega t - \frac{1}{C\omega}\sin\omega t\right)}{R^2 + \frac{1}{C^2\omega^2}} + \frac{E\epsilon^{-\frac{1}{RC}t}}{RC^2\omega^2\left(R^2 + \frac{1}{C^2\omega^2}\right)},$$

which may be written in this form:

$$i = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}}\left\{\cos(\omega t + \psi) + \frac{\sin\psi}{RC\omega}\epsilon^{-\frac{1}{RC}t}\right\}, \tag{30}$$

$$\tan\psi = \frac{1}{RC\omega}$$

the complete solution for the current in a condenser circuit; the first term the permanent current component, and the second term the transient component.

Circuit of Inductance L, Capacity C, and Resistance R.— For a circuit of inductance, capacity, and resistance in series, applied steady voltage, the circuit equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = E,$$

and

$$i = \frac{E}{Lp + R + \dfrac{1}{Cp}}. \tag{31}$$

The determinantal equation is

$$Z(p) = Lp + R + \frac{1}{Cp} = 0,$$

or

$$LCp^2 + RCp + 1 = 0,$$

a second-degree equation, the roots of which are:

$$\left.\begin{aligned} p_1 &= -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\alpha + j\beta, \\ p_2 &= -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\alpha - j\beta. \end{aligned}\right\} \tag{32}$$

We also have

$$\frac{\partial Z(p)}{\partial p} = L - \frac{1}{Cp^2}. \tag{33}$$

Introducing the values of p_1 and p_2 from (32) into (33), we get

$$\left.\begin{aligned} \frac{\partial Z(p)}{\partial p}_{p=p_1} &= L - \frac{1}{C(-\alpha + j\beta)^2}, \\ \frac{\partial Z(p)}{\partial p}_{p=p_2} &= L - \frac{1}{C(-\alpha - j\beta)^2}. \end{aligned}\right\} \tag{34}$$

For $p = 0$; $1/Cp = \infty$ and $Z(p) = \infty$, the steady-current component is zero, as is to be expected.

For the transient current we have, on substituting the values of p and $\partial Z(p)/\partial p$ from (33) and (34) into (13), the expansion formula

$$\begin{aligned} i = E\epsilon^{-\alpha t}\Bigg\{ &\frac{\epsilon^{j\beta t}}{(-\alpha + j\beta)\left(L - \dfrac{1}{C(-\alpha + j\beta)^2}\right)} + \\ &\frac{\epsilon^{-j\beta t}}{(-\alpha - j\beta)\left(L - \dfrac{1}{C(-\alpha - j\beta)^2}\right)}\Bigg\}, \\ = E\epsilon^{-\alpha t}\Bigg\{ &\frac{\epsilon^{j\beta t}}{L(-\alpha + j\beta) - \dfrac{1}{C(-\alpha + j\beta)}} \\ &\frac{\epsilon^{-j\beta t}}{L(\alpha + j\beta) - \dfrac{1}{C(\alpha + j\beta)}}\Bigg\}. \end{aligned}$$

By a simple algebraic transformation, the above reduces to

$$i = E\epsilon^{-\alpha t}\left\{\frac{\epsilon^{j\beta t}}{2Lj\beta} - \frac{\epsilon^{-j\beta t}}{2Lj\beta}\right\},$$

$$= \frac{E\epsilon^{-\alpha t}\sin\beta t}{L\beta}. \qquad (35)$$

a damped oscillatory current, the damping factor is

$$\alpha = \frac{R}{2L},$$

and the frequency is given by

$$f = \frac{\beta}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \qquad (36)$$

If we neglect the resistance term in the expression for the frequency, (35) simplifies to

$$i = \frac{E\epsilon^{-\alpha t}\sin\beta t}{L\dfrac{1}{\sqrt{LC}}} = E\sqrt{\frac{C}{L}}\epsilon^{-\alpha t}\sin\beta t. \qquad (37)$$

The above results hold also for the case $R^2/4L^2 > 1/LC$. When, however, $1/LC = R^2/4L^2$, the critical case, the two roots p_1 and p_2 are equal and the expansion formula in the form given is no longer applicable. We may arrive, however, at the expression for the current for this case in the following way: When p_1 and p_2 approach equality, β approaches zero value. For very small difference between p_1 and p_2, β is very small, $\sin\beta t = \beta t$, and (35) reduces to

$$i = \frac{Et\epsilon^{-\alpha t}}{L}, \qquad (38)$$

independent of β, and holds also for the limiting value

$$p_1 = p_2;\ \beta = 0.$$

The expression for i for the critical case given by (38) can be arrived at in another way. Put the circuit equation (31) in this form:

$$i = \frac{ECp}{LC\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)},$$

and for

$$\frac{1}{LC} = \frac{R^2}{4L^2} = \alpha^2,$$

$$i = \frac{Ep}{L(p+\alpha)^2}. \tag{39}$$

We note, however, that

$$\frac{p}{(p+\alpha)^2} = -\frac{d}{d\alpha}\left(\frac{p}{p+\alpha}\right),$$

and

$$\frac{p}{p+\alpha} = \frac{1}{1+\frac{\alpha}{p}} = \epsilon^{-\alpha t} \text{ (see equation (12), Chap. I).}$$

Hence,

$$\frac{p}{(p+\alpha)^2} = -\frac{d}{d\alpha}\epsilon^{-\alpha t} = t\epsilon^{-\alpha t}. \tag{40}$$

Introducing this value in (39), we obtain

$$i = \frac{Et}{L}\epsilon^{-\alpha t},$$

which is the same as (38).

For the case of an alternating e.m.f. impressed on the circuit, the modified expansion formula (20) or (21) is to be applied. It is somewhat simpler to use formula (20) in this case.

The values of p_1, p_2, and $\frac{\partial Z(p)}{\partial p}$ are, of course, the same, given by (32) and (34). The expressions for $\partial Z(p)/\partial p$ may be put in a simpler form:

$$\begin{aligned}\frac{\partial Z(p)}{\partial p}_{p=p_1} &= L - \frac{1}{C(-\alpha+j\beta)^2} = \frac{LC(\alpha^2-\beta^2-2j\alpha\beta)-1}{C(-\alpha+j\beta)^2} \\ &= \frac{2LC\left(\frac{R^2}{4L^2}-\frac{1}{LC}-j\alpha\beta\right)}{C(-\alpha+j\beta)^2} \\ &= \frac{2Lj\beta}{-\alpha+j\beta} = \frac{2L\beta}{\beta+j\alpha}.\end{aligned}$$

Similarly,

$$\frac{\partial Z(p)}{\partial p}_{p=p_2} = \frac{2Lj\beta}{\alpha+j\beta} = \frac{2L\beta}{\beta-j\alpha}.$$

Introducing these values in (20), we obtain

$$i = \frac{E\epsilon^{j\omega t}}{Lj\omega + R + \frac{1}{Cj\omega}} - \frac{E\epsilon^{-\alpha t}}{2L\beta}\left\{\frac{\epsilon^{j\beta t}(\beta + j\alpha)}{j\omega + \alpha - j\beta} + \frac{(\beta - j\alpha)\epsilon^{-j\beta t}}{j\omega + \alpha + j\beta}\right\}. \quad (41)$$

The first right-hand term in the above expression represents the steady-state, periodic component; and the other two terms, the transient component. Combining the two terms of the transient component, we get

$$i_t = \frac{E\epsilon^{-\alpha t}}{2L\beta}$$

$$\left\{\frac{(j\beta\omega + j\beta^2 - \omega\alpha + j\alpha^2)\epsilon^{j\beta t} + (j\beta\omega - j\beta^2 + \omega\alpha - j\alpha^2)\epsilon^{-j\beta t}}{\beta^2 + \alpha^2 - \omega^2 + 2j^{\omega\alpha}}\right\}.$$

This reduces to

$$i_t = \frac{E\epsilon^{-\alpha t}}{L\beta}\frac{\{j\beta\omega \cos \beta t - (\beta^2 + \alpha^2 + j\omega\alpha) \sin \beta t\}}{\beta^2 + \alpha^2 - \omega^2 + 2j\omega\alpha} \quad (42)$$

The real part of (42) is

$$i = \frac{E\epsilon^{-\alpha t}}{L\beta}\frac{\{2\alpha\beta\omega^2 \cos \beta t - [(\alpha^2 + \beta^2)^2 + \omega^2(\alpha^2 - \beta^2)] \sin \beta t\}}{(\beta^2 + \alpha^2 - \omega^2)^2 + 4\alpha^2\omega^2},$$

which may be put in the simpler form

$$i_t = -\frac{E\epsilon^{-\alpha t}}{L\beta}\frac{(\alpha^2 + \beta^2) \sin (\beta t - \psi)}{\sqrt{(\beta^2 + \alpha^2 - \omega^2)^2 + 4\alpha^2\omega^2}}, \quad (43)$$

$$\tan \psi = \frac{2\alpha\beta\omega}{(\alpha^2 + \beta^2)^2 + \omega^2(\alpha^2 - \beta^2)}.$$

Taking the real part of the first right-hand term of (41) and combining with (43), we have the complete expression for the current:

$$i = \frac{E \cos (\omega t - \varphi)}{\sqrt{\left(L\omega - \frac{1}{C\omega}\right)^2 + R^2}} + \frac{E\epsilon^{-\alpha t}}{L\beta}\frac{(\alpha^2 + \beta^2) \sin (\beta t - \psi)}{\sqrt{(\beta^2 + \alpha^2 - \omega^2)^2 + 4\alpha^2\omega^2}}. \quad (44)$$

For $\omega = 0$, the steady-current component is zero, and the transient component reduces to

$$i = \frac{E\epsilon^{-\alpha t}}{L\beta} \sin \beta t.$$

the same as (35).

We have so far considered the application of the expansion theorem for the determination of the current in a circuit. The method, however, is not at all limited to that particular variable. We could have, for instance, in the preceding illustration, developed the solution for the voltage across the condenser, by the application of the expansion theorem to the expression for the voltage across the condenser,

$$V = \frac{E}{LCp^2 + RCp + 1}.$$

In this simple case, there is no particular advantage in developing the solution for the voltage in preference to the current. In the case, however, of problems relating to circuits of distributed electrical constants, it is sometimes an advantage to apply the expansion theorem to determine the voltage instead of the current at some point in the circuit and derive the solution for the current from the voltage by the circuital relation between them.

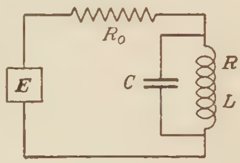

Fig. 4.

Divided Circuits.—A circuit comprising an inductance and a capacity in parallel, and this parallel circuit in series with a resistance, the circuit arrangement is shown in Fig. 4.

If we designate the impedance of the inductance branch by z_1 and the capacity branch by z_2, we have the circuit equations

$$\left.\begin{aligned} R_0i_0 + z_1i_1 &= E, \\ R_0i_0 + z_2i_2 &= E, \end{aligned}\right\} \tag{45}$$

and the auxiliary relation

$$i_0 = i_1 + i_2.$$

From these, the expressions for i_1 and i_2 are readily derived, thus:

$$\left.\begin{aligned} i_1 &= \frac{Ez_2}{R_0(z_1 + z_2) + z_1z_2}, \\ i_2 &= \frac{Ez_1}{R_0(z_1 + z_2) + z_1z_2}. \end{aligned}\right\} \tag{46}$$

In one case, the determinantal equation is

$$Z(p) = \frac{R_0(z_1 + z_2) + z_1 z_2}{z_2} = 0,$$

and in the other case,

$$Z(p) = \frac{R_0(z_1 + z_2) + z_1 z_2}{z_1} = 0. \tag{47}$$

The roots of the determinantal equation are the same in both cases, determined from the equation

$$R_0(z_1 + z_2) + z_1 z_2 = 0,$$

$$R_0\left(L_1 p + R_1 + \frac{1}{C_2 p}\right) + \frac{1}{C_2 p}(L_1 p + R_1) = 0,$$

which yields the following values of p:

$$p = -\alpha \pm j\beta$$

where

$$\alpha = \frac{R_1}{2L_1} + \frac{1}{2R_0 C_2},$$

$$\beta = \sqrt{\frac{R_0 + R_1}{R_0 L_1 C_2} - \left(\frac{R_1}{2L_1} + \frac{1}{2R_0 C_2}\right)^2} \tag{48}$$

For

$$p = 0; \quad z_2 = \frac{1}{C_2 p} = \infty.$$

Hence,

$$\text{for } i_1; \quad Z(p)_{p=0} = R_0 + R_1$$

$$\text{for } i_2; \quad Z(p)_{p=0} = \infty.$$

We have also for i_1,

$$\frac{\partial Z(p)}{\partial p} = \frac{\partial}{\partial p}\left\{R_0\left(Lp + R_1 + \frac{1}{C_2 p}\right) + \frac{1}{C_2 p}(L_1 p + R_1)\right\} C_2 p$$

$$= C_2 p\left\{R_0 L_1 - \frac{R_0}{C_2 p^2} - \frac{R_1}{C_2 p^2}\right\}, \tag{49}$$

remembering that the bracket term is zero for the values of p which are the roots of the determinant equation, and these are the values of p to be used in the expansion formula. For i_2,

$$\frac{\partial Z(p)}{\partial p} = \frac{\partial}{\partial p}\left\{\frac{R_0\left(L_1 p + R_1 + \frac{1}{C_2 p}\right) + \frac{1}{C_2 p}(L_1 p + R_1)}{L_1 p + R_1}\right\},$$

$$= \frac{R_0 L_1 C_2 p^2 - (R_0 + R_1)}{C_2 p^2 (L_1 p + R_1)}. \tag{50}$$

Introducing these values in the expansion formula, we obtain the following expressions for the currents in the two branches:

$$\left.\begin{aligned} i_1 &= \frac{E}{R_0 + R_1} + \frac{E\epsilon^{(-\alpha \pm j\beta)t}}{R_0L_1C_2p^2 - (R_0 + R_1)}, \\ i_2 &= \frac{EC_2p(L_1p + R_1)}{R_0L_1C_2p^2 - (R_0 + R_1)}\epsilon^{(-\alpha \pm j\beta)t}. \end{aligned}\right\} \quad (51)$$

Substituting the value of p given by (48), the complete solutions for the current distributions in the circuits are obtained.

The current in the main circuit is

$$i_0 = i_1 + i_2 = \frac{E}{R_0 + R_1} + E\frac{\{1 + L_1C_2p^2 + R_1C_2p\}}{R_0L_1C_2p^2 - (R_0 + R_1)}\epsilon^{(-\alpha \pm j\beta)t}. \quad (52)$$

When $C = 0$, that is, the condenser branch open, the expression for the current i_0 should reduce to that of the current in an inductive circuit whose resistance is the sum of the resistances R_0 and R_1. This it actually does, as is seen from the following consideration: For C very small,

$$\alpha = \frac{1}{2R_0C_2} + \frac{R_1}{2L_1}.$$

$$\begin{aligned} \beta &= \sqrt{\frac{1}{L_1C_2} - \left(\frac{1}{2C_2R_0} + \frac{R_1}{2L_1}\right)^2} \\ &= \sqrt{\frac{1}{L_1C_2} - \frac{1}{4C_2^2R_0^2} + \frac{R_1}{2L_1C_2R_0}} \left(\text{neglecting } \frac{R_1^2}{4L_1^2}\right) \\ &= j\frac{1}{2C_2R_0}\left\{1 - \frac{2C_2R_0^2}{L_1} - \frac{R_1C_2R_0}{L_1}\right\}, \text{ approximately} \end{aligned}$$

and

$$\begin{aligned} p = -\alpha - j\beta &= \frac{-1}{2R_0C_2} - \frac{R_1}{2L_1} + \frac{1}{2R_0C_2} - \frac{R_0}{L_1} - \frac{R_1}{2L_1} \\ &= -\frac{R_0 + R_1}{L_1}. \end{aligned} \quad (53)$$

Putting $C = 0$ in equation (52), it reduces to

$$i_0 = \frac{E}{R_0 + R_1} - \frac{E}{R_0 + R_1}\epsilon^{-\frac{R_0+R_1}{L}t}; \quad (54)$$

the expression for the current in an inductive circuit.

For $L_1 = \infty$, $R_1 = \infty$, the inductive branch open,

$$\alpha = \frac{R_1}{2L_1} + \frac{1}{2C_2R_0}$$

$$\beta = \sqrt{\frac{1}{L_1C_2} - \left(\frac{1}{2C_2R_0} - \frac{R_1}{2L_1}\right)^2}.$$

$$= j\left(\frac{1}{2C_2R_0} - \frac{R_1}{2L_1}\right);\ \frac{1}{L_1C_2} = 0$$

$$p = -\alpha + j\beta = -\frac{R_1}{2L_1} - \frac{1}{2C_2R_0} - \frac{1}{2C_2R_0} + \frac{R_1}{2L_1}$$

$$= -\frac{1}{C_2R_0}. \tag{55}$$

For

$$R_1 = L_1 = \infty;\ \frac{E}{R_0 + R_1} = 0,$$

and

$$\frac{E(1 + L_1C_2p^2 + R_1C_2p)}{R_0L_1C_2p^2 - (R_0 + R_1)} = \frac{E(L_1C_2p^2 + R_1C_2p)}{R_0L_1C_2p^2 - R_1}$$

Substituting the value of p from (55), it reduces to

$$\frac{E\left(\frac{L_1}{C_2R_0^2} - \frac{R_1}{R_0}\right)}{\frac{L_1}{C_2R_0} - R_1} = \frac{E}{R_0}$$

Introducing these values in equation (52), we have, finally, for the condition $L_1 = R_1 = \infty$,

$$i_0 = \frac{E}{R_0}\epsilon^{-\frac{1}{R_0C_2}}, \tag{56}$$

the expression for the charging current in a circuit of resistance and capacity.

Coupled Inductive Circuits.—Two circuits each consisting of an inductance and a resistance, coupled magnetically, a steady voltage applied to one circuit. The circuit equations are

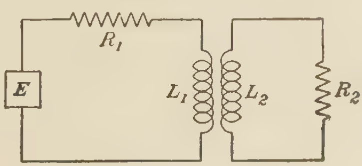

FIG. 5.

$$\left.\begin{aligned}(L_1p + R_1)i_1 + Mpi_2 &= E,\\ (L_2p + R_2)i_2 + Mpi_1 &= 0,\end{aligned}\right\} \tag{57}$$

from which the expressions for i_1 and i_2 are readily obtained.

$$\left.\begin{aligned}i_1 &= \frac{E(L_2p + R_2)}{(L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2},\\ i_2 &= \frac{-EMp}{(L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2}.\end{aligned}\right\} \tag{58}$$

To develop the solutions for the currents in the circuits from the above expressions, it is necessary, of course, to obtain the roots of the determinantal equation,

$$Z(p) = (L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2 = 0.$$

The roots are readily determined as follows:

$$\left.\begin{matrix}p_1\\ p_2\end{matrix}\right\} = \frac{-(R_1L_2 + R_2L_1) \pm \sqrt{(R_1L_2 - R_2L_1)^2 + 4R_1R_2M^2}}{2(L_1L_2 - M^2)}. \tag{59}$$

This may be put in a more convenient form,

$$\left.\begin{matrix}p_1\\ p_2\end{matrix}\right\} = \frac{-(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2K^2}}{1 - K^2}, \tag{60}$$

where

$$\alpha_1 = \frac{R_1}{2L_1};\ \alpha_2 = \frac{R_2}{2L_2};\ K^2 = \frac{M^2}{L_1L_2}.$$

We also have

$$\begin{aligned}\frac{\partial Z(p)}{\partial p} &= 2p(L_1L_2 - M^2) + (R_1L_2 + R_2L_1)\\ &= 2L_1L_2\{(1 - K^2)p + \alpha_1 + \alpha_2\}\\ &= \pm 2L_1L_2\sqrt{(\alpha_1 - \alpha_2)^2 + 4K^2\alpha_1\alpha_2}.\end{aligned} \tag{61}$$

For

$$p = 0.$$

$$Z(p)_{p=0} = R_1 \text{ for the primary circuit,}$$

and

$$Z(p)_{p=0} = 0 \text{ for the secondary circuit.}$$

which is obvious from equations (58).

Substituting these values in the expansion formula (13), we obtain the following expressions for the currents in the two circuits:

$$\begin{aligned}i_1 &= \frac{E}{R_1} + \frac{E}{2L_1L_2\sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2K^2}}\left\{\frac{L_2p_1 + R_2}{p_1}\epsilon^{p_1t} - \frac{(L_2p_2 + R_2)}{p_2}\epsilon^{p_2t}\right\}.\\ i_2 &= \frac{-EM}{2L_1L_2\sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2K^2}}\{\epsilon^{p_1t} - \epsilon^{p_2t}\}.\end{aligned} \tag{62}$$

The values of p_1 and p_2 are given by (60).

For $t = 0$ the bracket term in the expression for i_1 reduces to the following:

$$\frac{L_2p_1 + R_2}{p_1} - \frac{L_2p_2 + R_2}{p_2} = \frac{R_2(p_2 - p_1)}{p_1p_2}.$$

Introducing the values of p_1 and p_2 from (60) gives us

$$\frac{L_2p_1 + R_2}{p_1} - \frac{L_2p_2 + R_2}{p_2} =$$

$$\frac{-2R_2\sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2K^2}}{(1 - K^2)\left\{\frac{(\alpha_1 + \alpha_2)^2 - (\alpha_1 - \alpha_2)^2 - 4\alpha_1\alpha_2K^2}{(1 - K^2)^2}\right\}}$$

$$= \frac{2R_2\sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2K^2}}{4\alpha_1\alpha_2}.$$

Substituting this in the first equation (62), we have, for $t = 0$,

$$i_1 = \frac{E}{R_1} - \frac{2R_2E}{8\alpha_1\alpha_2L_1L_2} = \frac{E}{R_1} - \frac{E}{R_1} = 0.$$

The current in the secondary circuit is obviously zero for $t = 0$, the bracket term reduces to zero.

Coupled Oscillatory Circuits.—For two circuits each consisting of an inductance, capacity, and resistance, the circuit equations are

$$\left.\begin{aligned}\left(L_1p + R_1 + \frac{1}{C_1p}\right)i_1 + Mpi_2 &= E,\\ \left(L_2p + R_2 + \frac{1}{C_2p}\right)i_2 + Mpi_1 &= 0.\end{aligned}\right\} \qquad (63)$$

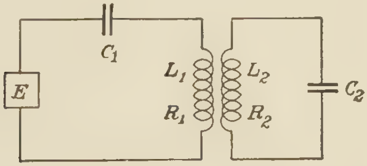

FIG. 6.

Solving for i_1 and i_2,

$$\left.\begin{aligned} i_1 &= \frac{E\left(L_2p + R_2 + \frac{1}{C_2p}\right)}{\left(L_1p + R_1 + \frac{1}{C_1p}\right)\left(L_2p + R_2 + \frac{1}{C_2p}\right) - M^2p^2},\\ i_2 &= \frac{-EMp}{\left(L_1p + R_1 + \frac{1}{C_1p}\right)\left(L_2p + R_2 + \frac{1}{C_2p}\right) - M^2p^2}.\end{aligned}\right\} \qquad (64)$$

It is observed that in the above expressions for either i_1 or i_2, the determinantal equation is a fourth-degree equation. To develop the complete solutions from the above expressions by the application of the expansion theorem requires the determination of the four roots of a fourth-degree equation, which would introduce considerable mathematical complexity. This problem is considered fully in Chap. III, Filter Circuits, where questions relating to multiperiodic circuit systems are considered. We shall limit the discussion here to circuits of negligible resistance, in which case the determinantal equation is reduced to a biquadratic equation, the roots of which are readily determined.

For $R = 0$, the circuit equations (64) reduce to the following:

$$\left.\begin{aligned} i_1 &= \frac{E\left(L_2p + \dfrac{1}{C_2p}\right)C_1C_2p^2}{C_1C_2(L_1L_2 - M^2)p^4 + (L_2C_2 + L_1C_1)p^2 + 1}, \\ i_2 &= \frac{-EMC_1C_2p^3}{C_1C_2(L_1L_2 - M^2)p^4 + (L_2C_2 + L_1C_1)p^2 + 1}. \end{aligned}\right\} \qquad (65)$$

For either circuit, the determinantal equation is

$$Z(p) = (L_1L_2 - M^2)C_1C_2p^4 + (L_2C_2 + L_1C_1)p^2 + 1 = 0. \qquad (66)$$

The roots of this equation are given by

$$p_s = \pm\sqrt{\frac{-(L_1C_1 + L_2C_2) \pm \sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}{2(L_1L_2 - M^2)C_1C_2}}. \qquad (67)$$

The four values of p_s given by (67) may be put in this form:

$$\left.\begin{aligned} p_1 &= j\beta_1; \quad p_2 = -j\beta_1; \\ p_3 &= j\beta_2; \quad p_4 = -j\beta_2, \end{aligned}\right\} \qquad (68)$$

where

$$\left.\begin{aligned} \beta_1 &= \sqrt{\frac{(L_1C_1 + L_2C_2) + \sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}{2(L_1L_2 - M^2)C_1C_2}}, \\ \beta_2 &= \sqrt{\frac{(L_1C_1 + L_2C_2) - \sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}{2(L_1L_2 - M^2)C_1C_2}}. \end{aligned}\right\} \qquad (69)$$

For $L_1C_1 = L_2C_2$ the above reduces to

$$\left.\begin{aligned} \beta_1 &= \frac{1}{\sqrt{LC(1 + K)}}, \\ \beta_2 &= \frac{1}{\sqrt{LC(1 - K)}}. \end{aligned}\right\} \qquad (70)$$

where

$$K = \frac{M}{\sqrt{L_1L_2}}$$

The values of $\partial Z(p)/\partial p$ for the different values of p are readily obtained from (66).

$$\frac{\partial Z(p)}{\partial p} = 2p\{2(L_1L_2 - M^2)p^2 + (L_2C_2 + L_1C_1)\},$$

and substituting for p^2 the value given by (67), it simplifies to

$$\frac{\partial Z(p)}{\partial p} = \pm 2p\sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}. \qquad (71)$$

For $p = 0$, $Z(p) = \infty$ obviously. The steady-current component in either the primary or the secondary circuit is zero, which we should expect. By substituting the values of p and $\partial Z(p)/\partial p$ in the expansion formula (13), we obtain the complete solutions for the currents in either the primary or secondary circuit. Consider the secondary circuit:

$$\begin{aligned} i_2 &= -EMC_1C_2\sum \frac{p_s^3\epsilon^{p_st}}{\pm 2p_s^2\sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}. \\ &= \frac{-EMC_1C_2}{2\sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}\sum \pm\, p_s\epsilon^{p_st}, \\ &= \frac{-EMC_1C_2}{2\sqrt{L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}\{j\beta_1\epsilon^{j\beta_1t} - j\beta_1\epsilon^{-j\beta_1t} - \\ &\qquad\qquad j\beta_2\epsilon^{j\beta_2t} + j\beta_2\epsilon^{-j\beta_2t}\} \\ &= \frac{EMC_1C_2}{2\sqrt{(L_1C_1 - L_2C_2)^2 + 4M^2C_1C_2}}\{\beta_1 \sin \beta_1t - \beta_2 \sin \beta_2t\}\cdot \qquad (72) \end{aligned}$$

The current consists of two components, oscillatory currents of different frequencies; $f_1 = \beta_1/2\pi$ and $f_2 = \beta_2/2\pi$. When the circuits are syntonized, that is, $L_1C_1 = L_2C_2$, the above reduces to

$$i_2 = \frac{E}{2}\sqrt{C_1C_2}\{\beta_1 \sin \beta_1t - \beta_2 \sin \beta_2t\}, \qquad (73)$$

and the values of β_1 and β_2 are given by (70).

Subsidence of Current in Circuits.—The problems we have considered thus far all relate to this question: A voltage, steady or alternating, is suddenly applied to a circuit system; what is the character of the current at any time after the application of the voltage? We found that, in every case, the formula for the

current, which is, indeed, the expansion formula, consists of two parts, one of which is the permanent-current component, and the other the transient-current component. In some cases, the permanent current component is zero, as in the case of applied steady voltage to a condenser circuit. The transient current is expressed by the summation term in the expansion formula, the number of terms occurring in the summation depending on the character of the circuit system.

The expansion theorem, however, is not limited in its application to the determination of the transient-current rise in a circuit after the closing of the switch or the application of the voltage; it also gives correctly, except for a reversal of sign, the subsidence of the current in a circuit system when the voltage source is suddenly short-circuited. This can be demonstrated in this way: Suppose the circuit was closed for a sufficiently long time to insure the establishment of the steady-state condition, the transient component completely wiped out, which, in most practical cases, obtains within a short time after the closing of the switch; the current in the circuit is then expressed by the first term of the expansion formula, thus:

$$i = \frac{E}{Z(p)_{p=0}},$$

or

$$i = \frac{E}{Z(p)_{p=j\omega}},$$

depending on whether the voltage is steady or alternating. Now suppose that, instead of short-circuiting the voltage source, we introduce in the circuit another voltage of exactly the same character and magnitude but of opposite sign; the effect is the same as if the original voltage were removed or short-circuited. We have then, in this case, two similar voltages of opposite sign acting simultaneously on the circuit. The original voltage producing a current in the circuit

$$i = \frac{E}{Z(p)_{p=0}},$$

and the suddenly applied additional voltage producing a current in the circuit in accordance with the expansion formula

$$i = -\frac{E}{Z(p)_{p=0}} - E\sum \frac{\epsilon^{pt}}{p\frac{\partial Z(p)}{\partial p}}. \tag{74}$$

The resultant current in the circuit is the sum of the two currents, that is,

$$i = -E \sum_{p} \frac{\epsilon^{pt}}{p \frac{\partial Z(p)}{\partial p}}, \tag{75}$$

which is the formula for the subsidence of the current in a circuit on short-circuiting the voltage. The summation term, therefore, of the expansion formula gives also the current subsidence in a circuit on short-circuiting the voltage.

CHAPTER III

ELECTRIC FILTER CIRCUITS

The term *electric filter* is used to designate a circuit system consisting of a number of recurrent sections connected in tandem, each section comprising inductance and capacity in various combinations depending upon the requirements for which the system is designed. Actually, a circuit system of this kind is nothing more, of course, than a circuit combination comprising several circuits coupled electrically and is characterized by a multiplicity of degrees of freedom depending on the number of sections or individual circuits in the system. That is, the number of free vibrations of the circuit system is determined by the number of individual circuits. Obviously, the system will respond resonantly to the frequencies corresponding to the frequencies of the free vibrations of the system. It permits to pass freely several frequencies within a predetermined range governed by the

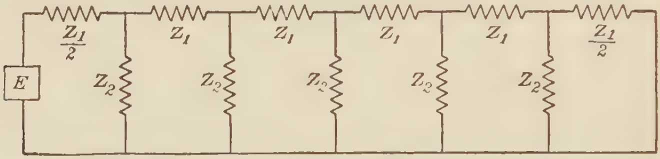

FIG. 7.

number of sections and the choice of the electrical constants of each section. A better term to designate a circuit system of this kind would have been *multiple periodic circuit system.* Since, however, it is now widely known under the designation *filter circuit,* the term is retained here.

Consider a circuit system diagrammatically shown in Fig. 7. The series impedances are designated by z_1, and the shunt impedances by z_2.

Either z_1 or z_2 may be any combination of inductance and capacity. An e.m.f. is applied to the first section through an impedance $z_1/2$, and the last section is closed through a series impedance $z_1/2$. The mathematical analysis is much simplified by this choice of terminations. We shall also designate the

currents in the series elements by subscripts corresponding to the number of the section. For any section, say the qth, except the first and last sections, the voltage drop in the circuits in accordance with Kirchoff's law is given by the following equation:

$$\left.\begin{aligned} z_1 i_q + z_2(i_q - i_{q+1}) - z_2(i_{q-1} - i_q) &= 0, \\ \text{or} \qquad (z_1 + 2z_2)i_q - z_2(i_{q+1} + i_{q-1}) &= 0. \end{aligned}\right\} \tag{1}$$

We assume a solution of the form

$$i_q = A\epsilon^{q\gamma} + B\epsilon^{-q\gamma}, \tag{2}$$

where A and B are independent of q. On substitution and after a simple rearrangement, we get

$$\left(2 + \frac{z_1}{z_2} - \epsilon^{\gamma} - \epsilon^{-\gamma}\right)(A\epsilon^{q\gamma} + B\epsilon^{-q\gamma}) = 0. \tag{3}$$

Clearly, (2) is a solution of (1), provided the value of γ is chosen to satisfy the relation

$$\epsilon^{\gamma} + \epsilon^{-\gamma} = 2 + \frac{z_1}{z_2},$$

or

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2}. \tag{4}$$

The constants A and B are determined from the terminal conditions, that is, the circuit equations of the first and last sections. For the first, that is, the zero section, we have

$$\frac{z_1}{2}i_0 + z_2(i_0 - i_1) = E, \tag{5}$$

and for the last, the nth section,

$$\frac{z_1}{2}i_n - z_2(i_{n-1} - i_n) = 0. \tag{6}$$

Using the values of i given by (2), we obtain the following two equations from which the constants A and B are determined:

$$\left.\begin{aligned} \left(\frac{z_1}{2} + z_2 - z_2\epsilon^{\gamma}\right)A + \left(\frac{z_1}{2} + z_2 - z_2\epsilon^{-\gamma}\right)B &= E, \\ \left(\frac{z_1}{2} + z_2 - z_2\epsilon^{-\gamma}\right)A^{n\gamma} + \left(\frac{z_1}{2} + z_2 - z_2\epsilon^{\gamma}\right)B\epsilon^{-n\gamma} &= 0. \end{aligned}\right\} \tag{7}$$

Replacing $\frac{z_1}{2} + z_2$ by $\frac{z_2}{2}(\epsilon^{\gamma} + \epsilon^{-\gamma})$, the relation given by (4), and rearranging, we obtain the following:

$$\left.\begin{aligned} A - B &= \frac{-2E}{z_2(\epsilon^{\gamma} - \epsilon^{-\gamma})}, \\ A\epsilon^{n\gamma} - B\epsilon^{-n\gamma} &= 0. \end{aligned}\right\} \tag{8}$$

From these, we obtain

$$\left.\begin{aligned} A &= \frac{2E\epsilon^{-n\gamma}}{z_2(\epsilon^{\gamma} - \epsilon^{-\gamma})(\epsilon^{n\gamma} - \epsilon^{-n\gamma})}, \\ B &= \frac{2E\epsilon^{n\gamma}}{z_2(\epsilon^{\gamma} - \epsilon^{-\gamma})(\epsilon^{n\gamma} - \epsilon^{-n\gamma})}. \end{aligned}\right\} \tag{9}$$

Substituting these values in (2), we finally get the complete expression for the current in any section, as follows:

$$\dot{i}_q = \frac{2E\{\epsilon^{(n-q)\gamma} + \epsilon^{-(n-q)\gamma}\}}{z_2(\epsilon^{\gamma} - \epsilon^{-\gamma})(\epsilon^{n\gamma} - \epsilon^{-n\gamma})}, \tag{10}$$

or

$$\dot{i}_q = \frac{E \cosh (n - q)\gamma}{z_2 \sinh \gamma \sinh n\gamma}. \tag{11}$$

For the last, the nth section, $q = n$,

$$\dot{i}_n = \frac{E}{z_2 \sinh \gamma \sinh n\gamma}. \tag{12}$$

The complete solution can be readily developed from (11) and (12) by the application of the expansion theorem. In practical work, the main concern is the output current, the current in the last section given by (12). The solution and result will depend on the particular combination of inductance and capacity in the series and shunt elements z_1 and z_2. We shall consider several typical arrangements for steady and alternating voltages.

Series Element $z_1 = Lp + R$ and Shunt Element $z_2 = 1/Cp$ Steady Voltage E.—The determinantal equation is

$$Z(p) = z_2 \sinh \gamma \sinh n\gamma = 0, \tag{13}$$

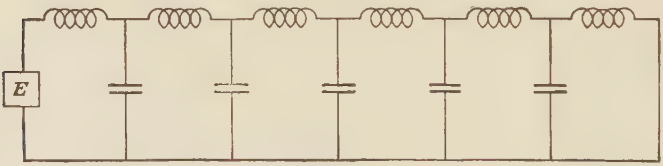

FIG. 8.

which gives

$$n\gamma = js\pi;\ s = 0, 1, 2, 3, \cdots n,$$

$$\gamma = j\frac{s\pi}{n}. \tag{14}$$

The zero value of s is to be disregarded, for the reason that this would not satisfy the relation given by (4), leading to the result $z_1/z_2 = 0$, an impossible condition. Also, all values of s greater

than n will give only repetitions of preceding values. Hence, values of s from 1 to n are to be used only. The values of p corresponding to the roots of the determinantal equation (14) can be determined from the relation given by (4). In some cases, z_2, which is a factor in the determinantal equation, may be a combination of inductance and capacity to yield additional roots which should be taken into consideration in the complete solution. We shall have occasion to discuss this farther on.

For the case under consideration, $z_1 = Lp + R$ and $z_2 = 1/Cp$, we have, by (4),

$$\cosh \gamma = \cos \frac{s\pi}{n} = 1 + \frac{1}{2}Cp(Lp + R), \tag{15}$$

which, on solving for p, gives

$$p_s = -\alpha + j\beta_s, \tag{16}$$

$$\alpha = \frac{R}{2L}; \beta_s = \sqrt{\frac{2}{LC}\left(1 - \cos \frac{s\pi}{n}\right) - \frac{R^2}{4L^2}}, \tag{17}$$

also

$$\frac{\partial Z(p)}{\partial p} = z_2 \sinh \gamma \cosh n\gamma \frac{\partial(n\gamma)}{\partial p}. \tag{18}$$

By (15), we have

$$\sinh \frac{\partial\gamma}{\partial p} = LCp + \frac{RC}{2},$$

and

$$\frac{\partial\gamma}{\partial p} = \frac{LCp + \frac{RC}{2}}{\sinh \gamma}. \tag{19}$$

Introducing the value of $\partial\gamma/\partial p$ from (19) into (18) gives

$$\begin{aligned} \frac{\partial Z(p)}{\partial p} &= nz_2\left(LCp + \frac{RC}{2}\right) \cos (s\pi) \\ &= nL\left(1 + \frac{\alpha}{p}\right) \cos (s\pi), \end{aligned} \tag{20}$$

for $p = 0$; $Z(p)$ as given by (13) is indeterminate for $\sinh \gamma$ and $\sinh n\gamma$ are separately equal to zero, and z_2 is infinity. We can arrive, however, at the value of $Z(p)_{p=0} = 0$ by taking its value as p approaches zero.

For small values of p,

$$\cosh \gamma = 1 + \frac{\gamma^2}{2} = 1 + \frac{Cp}{2}(Lp + R),$$

and

$$\gamma^2 = Cp(Lp + R).$$

$$z_2 \sinh \gamma \sinh n\gamma = \frac{n\gamma^2}{Cp} = \frac{nCp(Lp + R)}{Cp} = nR \text{ for } p = 0. \quad (21)$$

Introducing the values of p_s, $\partial Z(p)/\partial p$, and $Z(p)_{p=0}$ from (16), (20), and (21) in the expansion formula (13), Chap. II, we arrive at the solution for the current in the nth section as follows:

$$i_n = \frac{E}{nR} + E\sum_{s=1}^{s=n} \frac{\epsilon^{(-\alpha \pm j\beta_s)t}}{(-\alpha \pm j\beta_s)\left(1 + \dfrac{\alpha}{-\alpha \pm j\beta_s}\right)nL \cos (s\pi)}$$

$$= \frac{E}{nR} + \frac{E\epsilon^{-\alpha t}}{nL}\sum_{s=1}^{s=n} \frac{\epsilon^{\pm j\beta_s t}}{\pm j\beta_s \cos (s\pi)}. \quad (22)$$

Taking proper account of the double-sign terms, including both terms under each double sign, the above reduces to

$$i_n = \frac{E}{nR} + \frac{2E\epsilon^{-\alpha t}}{nL}\sum_{s=1}^{s=n} \frac{\sin \beta_s t}{\beta_s \cos (s\pi)}. \quad (23)$$

The value of β_s given by (17) is

$$\beta_s = \sqrt{\frac{2}{LC}\left(1 - \cos\frac{s\pi}{n}\right) - \alpha^2}.$$

If we neglect α^2 compared with $\dfrac{2}{LC}\left(1 - \cos \dfrac{s\pi}{n}\right)$, the above reduces to

$$\beta_s = \sqrt{\frac{2}{LC}\left(1 - \cos\frac{s\pi}{n}\right)} = \frac{2 \sin \dfrac{s\pi}{2n}}{\sqrt{LC}}, \quad (24)$$

and introducing this in (23), we get the following:

$$i_n = \frac{E}{nR} + \frac{E\epsilon^{-\alpha t}}{n}\sqrt{\frac{C}{L}}\sum \frac{\sin \beta_s t}{\sin \dfrac{s\pi}{2n} \cos (s\pi)},$$

$$= \frac{E}{nR} + \frac{E\epsilon^{-\alpha t}}{n}\sqrt{\frac{C}{L}}\sum_{s=1}^{s=n} \frac{(-1)^s \sin \beta_s t}{\sin \dfrac{s\pi}{2n}}. \quad (25)$$

The first right-hand term is the steady-current component given by the impressed voltage divided by the total resistance of all the series elements. The summation term gives the transient current consisting of several components, the number determined by the number of sections or meshes in the circuit system; each component an independent damped oscillatory current, all having the same damping but of different frequencies, which are given by

$$f_s = \frac{\beta_s}{2\pi} = \frac{\sin\frac{s\pi}{2n}}{\pi\sqrt{LC}}. \tag{26}$$

The highest frequency of the oscillations is when $s = n$

$$f_n = \frac{1}{\pi\sqrt{LC}},$$

the upper limit of the frequency range is independent of the number of sections used. The lower frequency limit is decreased as the number of sections used is increased. When the number of sections is very large, the lower frequency limit approaches zero value. This type of circuit is commonly called *low pass filter*. Actually, it is nothing more than a circuit system capable of responding resonantly to a number of frequencies whose upper limit is $\frac{1}{\pi\sqrt{LC}}$, and the lower limit may be made as small as desired by increasing the number of sections and suitably selecting the values of the inductance and capacity in the sections. The separation between adjacent frequencies is obviously less and less as the number of sections is increased. For very many sections, the system will be resonantly responsive to many frequencies and thus approach the condition of what is generally designated as *band tuning*. Theoretically, the condition of so-called *band tuning* is realized only when the number of sections is infinitely great; but practically, the condition is approximated with a reasonably large number of sections.

Alternating E.m.f.—For applied alternating voltage, of sine wave form of frequency $f = \omega/2\pi$, the modified expansion formula (20) of Chap. II is to be applied,

$$i = \frac{E\epsilon^{j\omega t}}{Z(p)_{p=j\omega}} + E\sum\frac{p\epsilon^{pt}}{(p^2+\omega^2)\frac{\partial Z(p)}{\partial p}}. \tag{27}$$

the first right-hand term giving the permanent-current, and the summation term the transient-current components.

Consider first the summation term. The values of p and $\partial Z(p)/\partial p$ have already been obtained (equations (16) and (20)); hence,

$$\sum \frac{p\epsilon^{pt}}{(p^2+\omega^2)\dfrac{\partial Z(p)}{\partial p}} =$$

$$\sum \frac{(-\alpha \pm j\beta_s)\epsilon^{(-\alpha \pm j\beta_s)t}}{\{(-\alpha \pm j\beta_s)^2+\omega^2\}nL\left(1+\dfrac{\alpha}{-\alpha \pm j\beta_s}\right)\cos(s\pi)},$$

$$= \epsilon^{-\alpha t}\sum \frac{(-\alpha \pm j\beta_s)^2\epsilon^{\pm j\beta t}}{\pm j\beta_s nL\{(-\alpha \pm j\beta_s)^2+\omega^2\}\cos(s\pi)}.$$

Taking both terms under each double sign, we have

$$\sum \frac{p\epsilon^{pt}}{(p^2+\omega^2)\dfrac{\partial Z(p)}{\partial p}} = \epsilon^{-\alpha t}\sum \frac{(-\alpha + j\beta_s)(\cos\beta_s t + j\sin\beta_s t)}{j\beta_s nL\{(-\alpha + j\beta_s)^2+\omega^2\}\cos(s\pi)}$$

$$- \frac{(-\alpha - j\beta_s)^2(\cos\beta_s t - j\sin\beta_s t)}{j\beta_s nL\{(-\alpha - j\beta_s)^2+\omega^2\}\cos(s\pi)}.$$

By simple algebraic transformation, it reduces to the following:

$$\sum \frac{p\epsilon^{pt}}{(p^2+\omega^2)\dfrac{\partial Z(p)}{\partial p}} =$$

$$\epsilon^{-\alpha t}\sum \frac{2\{(\alpha^2+\beta_s^2)^2+\omega^2(\alpha^2-\beta_s^2)\}\sin\beta_s t - 4\alpha\beta_s\omega\cos\beta_s t}{nL\beta_s\{(\alpha^2+\beta_s^2)^2+2\omega^2(\alpha^2-\beta_s^2)+\omega^4\}\cos(s\pi)}. \quad (28)$$

For $\alpha = 0$, it reduces to

$$\sum \frac{p\epsilon^{pt}}{(p^2+\omega^2)\dfrac{\partial Z(p)}{\partial p}} = 2\sum \frac{\beta_s\sin\beta_s t}{nL(\beta_s^2-\omega^2)\cos(s\pi)}. \quad (29)$$

Comparing (28) and (29) with (25), it is seen that the oscillation frequencies of the transient current are the same in both cases, independent of the character of the applied voltage. The amplitudes of the oscillations, however, are modified by and depend upon the frequency of the applied voltage. If the frequency of the applied voltage is the same as one of the frequencies of the free oscillations of the system; that is, ω is equal to one of the values of β, the amplitude of the transient component of this

frequency will be infinite when the resistance is of zero value, which is evident from (29). This is what we should expect, the system responding resonantly to this frequency.

It remains yet to investigate the steady-state component

$$\begin{aligned} i &= \frac{E\epsilon^{pt}}{Z(p)_{p=j\omega}} \\ &= \frac{E\epsilon^{j\omega t}}{z_2 \sinh \gamma \sinh n\gamma_{p=j\omega}}. \qquad (30) \end{aligned}$$

We have the expression for $\cosh \gamma$

$$\begin{aligned} \cosh \gamma &= 1 + \frac{Cp(Lp + R)}{2} \\ &= 1 + \frac{Cj\omega(Lj\omega + R)}{2}. \end{aligned}$$

From this it should be possible to express the denominator of (30) in terms of ω. It is clear, however, that for n circuits, this would lead to an expression in ω of the nth degree, a complicated expression which would not throw any light on the problem. Of more interest is to consider for which values of ω the system would be in resonance. The preceding discussion relating to the transient components would lead us to expect a resonance effect at the frequencies of the free oscillations of the system. This is actually the case, and it can be shown in this way. If we neglect the resistance, the denominator in (30) must have zero value for the resonance frequencies; we shall determine these frequencies and show that they are precisely the frequencies of the free oscillations of the system.

By a well-known trigonometrical transformation, we may express the sine functions in a product series, thus:

$$\sin nx = K \sin x\left(1 - \frac{\sin^2 x}{\sin^2 \frac{\pi}{n}}\right)\left(1 - \frac{\sin^2 x}{\sin^2 \frac{2\pi}{n}}\right)\left(1 - \frac{\sin^2 x}{\sin^2 \frac{3\pi}{n}}\right) \cdots \qquad (31)$$

n is an odd integer

Put

$$\begin{aligned} x = j\gamma;\ \sin nx &= \sin jn\gamma = j \sinh n\gamma, \\ \sin x &= \sin j\gamma = j \sinh \gamma, \end{aligned}$$

and (31) transforms to

$$\sinh n\gamma = K \sinh \gamma\left(1 + \frac{\sinh^2 \gamma}{\sin^2 \frac{\pi}{n}}\right)\left(1 + \frac{\sinh^2 \gamma}{\sin^2 \frac{2\pi}{n}}\right) \cdots \qquad (32)$$

By (15), putting $p = \omega$, and neglecting resistances,

$$\sinh^2 \gamma = \cosh^2 \gamma - 1 = \left(1 + \frac{LC\omega^2}{2}\right)^2 - 1 = LC\omega^2\left(\frac{LC\omega^2}{4} + 1\right).$$

Substituting in the first factor of the power series (32), we obtain

$$1 + \frac{\sinh^2 \gamma}{\sin^2 \frac{\pi}{n}} = \frac{\sin^2 \frac{\pi}{n} + \frac{L^2C^2\omega^2}{4} + LC\omega^2}{\sin^2 \frac{\pi}{n}}.$$

The value of ω for which this is zero is readily obtained by solving the above quadratic equation, which gives

$$\omega^2 = \frac{2\left\{-1 \pm \sqrt{1 - \sin^2 \frac{\pi}{n}}\right\}}{LC} = \frac{2\left\{-1 + \cos \frac{\pi}{n}\right\}}{LC},$$

and

$$j\omega = \sqrt{\frac{2\left(1 - \cos \frac{\pi}{n}\right)}{LC}} = \frac{2 \sin \frac{\pi}{n}}{\sqrt{LC}}. \tag{33}$$

In a similar way, the other factors of the power series give

$$\left.\begin{aligned} j\omega_2 &= \frac{2 \sin \frac{2\pi}{n}}{\sqrt{LC}}, \\ j\omega_3 &= \frac{2 \sin \frac{3\pi}{n}}{\sqrt{LC}}. \end{aligned}\right\} \tag{34}$$

These are precisely the same as the values of β_s, equation (26), which establishes the fact that the frequencies for which the system is in resonance are the frequencies of the free oscillations of the system, as we should expect. This is but another way of arriving at the same result, determining the frequencies of the free vibrations of the circuit system.

Series-elements Capacities; Shunt-elements Inductances.— A circuit system of several recurrent sections in which the series elements are capacities, and the shunt elements are inductances, is commonly designated *high pass filter*. A system of this kind responds resonantly to frequencies above a fixed lower limit determined by the choice of values of the inductances and capacities in each section, the upper frequency limit depending on the

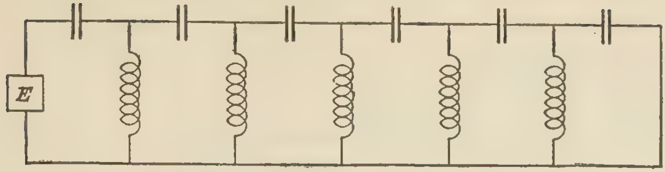

Fig. 9.

number of sections used. For this case,

$$\left.\begin{aligned} z_1 &= \frac{1}{Cp}, \\ z_2 &= Lp + R. \end{aligned}\right\} \tag{35}$$

Equation (12), the expression for the current in the nth section, is perfectly general, applicable for all cases, irrespective of the choice of inductance-capacity combination in either z_1 or z_2. In developing, however, the full solution from (12) by the application of the expansion theorem, cognizance must be taken of the changed conditions. We have, then, for this case also,

$$i_n = \frac{E}{z_2 \sinh \sinh n\gamma}, \tag{12 bis}$$

the determinantal equation as before,

$$Z(p) = z_2 \sinh \gamma \sinh n\gamma = 0, \tag{13 bis}$$

and

$$n\gamma = js\pi;\ s = 1, 2, 3, \cdot \cdot \cdot n$$

$$\gamma = \frac{js\pi}{n}. \tag{14 bis}$$

The frequencies corresponding to the roots of the determinantal equation are obtained from (4),

$$\cosh \gamma = 1 + \frac{z_1}{2z_2}, \tag{4 bis}$$

which for this circuit system gives

$$\cosh \gamma = \cos\left(\frac{s\pi}{n}\right) = 1 + \frac{1}{2Cp(Lp + R)}. \tag{36}$$

The values of p which satisfy the above equation are readily obtained, as follows:

$$p_s = -\alpha \pm j\beta_s,$$

where

$$\alpha = \frac{R}{2L};\ \beta_s = \sqrt{\frac{1}{2LC\left(1 - \cos\frac{s\pi}{n}\right)} - \frac{R^2}{4L^2}}. \tag{37}$$

We have, also,

$$\frac{\partial Z(p)}{\partial p} = z_2 \sinh \gamma \cosh n\gamma \frac{\partial (n\gamma)}{\partial p}. \tag{38}$$

By (4),

$$\sinh \gamma \frac{\partial \gamma}{\partial p} = -\frac{2Lp + R}{2Cp^2(Lp + R)^2},$$

and

$$z_2 \sinh \gamma \frac{\partial \gamma}{\partial p} = -\frac{(2Lp + R)}{2Cp^2(Lp + R)} = -\frac{\left(1 + \frac{\alpha}{p}\right)}{Cp^2\left(1 + \frac{2\alpha}{p}\right)}. \tag{39}$$

Introducing this in (38) gives

$$\begin{aligned}\frac{\partial Z(p)}{\partial p} &= -\frac{n\left(1 + \frac{\alpha}{p}\right)}{Cp^2\left(1 + \frac{2\alpha}{p}\right)} \cos (s\pi)\\ &= \frac{\mp nj\beta_s \cos (s\pi)}{C(\alpha \pm j\beta_s)(-\alpha \pm j\beta_s)^2}.\end{aligned} \tag{40}$$

For $p = 0$ in this case, $z_2 \sinh \gamma \sinh n\gamma = \infty$. The steady-current component is, therefore, zero, as we should expect, because the series elements are condensers.

Substituting the values of p_s and $\partial Z(p)/\partial p$ from (37) and (40) in the expansion formula, we obtain the developed expression for the current in the nth section,

$$\begin{aligned} i_n &= E \sum \frac{\epsilon^{(-\alpha \pm \beta_s)t}}{\dfrac{(-\alpha \pm j\beta_s)n(\pm j\beta_s)}{C(\alpha \pm j\beta_s)(-\alpha \pm j\beta_s)^2} \cos (s\pi)},\\ &= E\epsilon^{-\alpha t} \sum \frac{\epsilon^{\pm j\beta_s t}\, C(\alpha^2 - \beta_s^2)}{\pm nj\beta_s \cos (s\pi)}.\end{aligned} \tag{41}$$

Taking both terms under each double sign, the above reduces to

$$i_n = -2E\epsilon^{-\alpha t} \sum_{s=1}^{s=n} \frac{C(\alpha^2 - \beta_s^2) \sin \beta_s t}{n\beta_s \cos (s\pi)}. \tag{42}$$

For $\alpha = 0$, it reduces to

$$i_n = 2E \sum_{s=1}^{s=n} \frac{C\beta_s \sin \beta_s t}{n \cos (s\pi)}. \tag{43}$$

If we neglect the effect of the resistance in the expression for β_s in (37), it reduces to

$$\beta_s = \frac{1}{\sqrt{2LC\left(1 - \cos\frac{s\pi}{n}\right)}}$$

$$= \frac{1}{2 \sin \frac{s\pi}{2n}\sqrt{LC}}, \tag{44}$$

and introducing this in (43) gives

$$i_n = E\sum_{s=1}^{s=n}\sqrt{\frac{C}{L}}\frac{\sin \beta_s t}{n \sin \frac{s\pi}{n}}. \tag{45}$$

Comparing (25) and (45), it is seen that the transient currents in the two cases, *low pass filter* and *high pass filter*, are of exactly the same character, expressed by equations of the same identical form; the only difference is in the frequency characteristics of the two circuit systems. In this case, by (44),

$$f_s = \frac{\beta_s}{2\pi} = \frac{1}{4\pi \sin \frac{s\pi}{2n}\sqrt{LC}}.$$

For $s = n$,

$$\left.\begin{aligned} f_n &= \frac{1}{4\sqrt{LC}}. \\ \text{For } s = 1 \qquad & \\ f_1 &= \frac{1}{4\pi \sin \frac{\pi}{2n}\sqrt{LC}}. \end{aligned}\right\} \tag{46}$$

The frequency range is between these limits, the lowest frequency is $\frac{1}{4\pi\sqrt{LC}}$ and the upper frequency limit is

$$\frac{1}{4\pi \sin \frac{\pi}{2n}\sqrt{LC}},$$

depending, therefore, on n; as n is increased, the upper frequency limit is raised, which may be made to approach infinity by making n, the number of sections, infinitely large. In any case, a circuit system of this kind is necessarily responsive resonantly to a number of frequencies confined within the range

between f_1 and f_n given by (46), and by making n sufficiently large, an approximation to a uniform response to all frequencies within that range is obtained; and the condition for so-called *band tuning* for all frequencies above a certain lower frequency limit is approximately realized.

The investigation was limited, in this case, to steady voltage which is believed to be quite sufficient to explain the behavior and characteristics of the system. The derivation of the expression for applied alternating voltage is readily obtained by the second expansion formula, following the method used in connection with the preceding case, the *low pass filter*.

We have so far considered the simplest cases, those of a single reactance element, inductance, or capacity in either the series or shunt branches of the circuit system. It is quite obvious, however, that a large variety of series shunt combinations of inductance and capacity, in either the series or shunt branches, is possible, each combination affording a distinct circuit system of definite characteristics of its own, particularly in the matter of the frequencies of the free vibration periods of the circuit system. It is not within the scope of this book to consider fully the theory of filter circuits; the discussion is given here more for the purpose of illustrating the application of the expansion theorem to the solution of problems in electric-circuit theory. We shall, however, discuss briefly a few typical cases, confining the discussion to the determination of the frequencies of the free vibration periods, the governing characteristic in differentiating one circuit system from another. The complete solution in any particular case can be readily derived by the application of the expansion theorem in a manner outlined in connection with the derivation of the developed solution in the preceding cases.

Series-elements Inductances, Shunt-elements Inductances, and Capacities in Parallel.—Disregarding resistances, we have, for this case,

$$\left.\begin{aligned} z_1 &= L_1p, \\ z_2 &= \frac{L_2p}{1+L_2C_2p^2}. \end{aligned}\right\} \tag{47}$$

The frequencies of the free vibrations of the system are determined, as in the previous cases, from the conditions:

$$\sinh n\gamma = 0;$$

$$n\gamma = js\pi;\ s = 1, 2, 3, \cdots n,$$

and

$$\gamma = j\frac{s\pi}{n},$$

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2},$$

which give for this arrangement

$$\cos\left(\frac{s\pi}{n}\right) = 1 + \frac{1}{2}\frac{L_1}{L_2}(1 + L_2C_2p^2). \tag{48}$$

From this, the values of p are readily determined,

$$p_s = \sqrt{\frac{1}{L_2C_2}\left\{\frac{2L_2}{L_1}\left(\cos\frac{s\pi}{n} - 1\right) - 1\right\}}$$

$$= j\sqrt{\frac{1}{L_2C_2}\left\{\frac{4L_2}{L_1}\sin^2\frac{s\pi}{2n} + 1\right\}}. \tag{49}$$

This determines the frequencies of the system,

$$f_s = \frac{1}{2\pi\sqrt{L_2C_2}}\sqrt{1 + \frac{4L_2}{L_1}\sin^2\frac{s\pi}{2n}}. \tag{50}$$

The extreme frequencies at either end are given by

$$f_1 = \frac{1}{2\pi\sqrt{L_2C_2}}\sqrt{1 + \frac{4L_2}{L_1}\sin^2\frac{\pi}{2n}}, \tag{51}$$

$$f_n = \frac{1}{2\pi\sqrt{L_2C_2}}\sqrt{1 + \frac{4L_2}{L_1}}.$$

When n is very large, many sections, sin $2\pi/2n$ approaches zero value, and f_1 approaches the value $1/2\pi\sqrt{L_2C_2}$. All the frequencies are then confined within the limits $1/2\pi\sqrt{L_2C_2}$ and $\frac{1}{2\pi\sqrt{L_2C_2}}\sqrt{1 + 4\frac{L_2}{L_1}}$. Obviously, by making L_2/L_1 small, the frequency range is made small, and thus a condition of a circuit system capable of responding resonantly to a number of frequencies, all confined within a narrow range, is obtained, and the condition of so called *band tuning* for a narrow band of frequencies is thus approximately realized.

Series-elements Capacities, Shunt-elements Inductances, and Capacities in Parallel.—For this case,

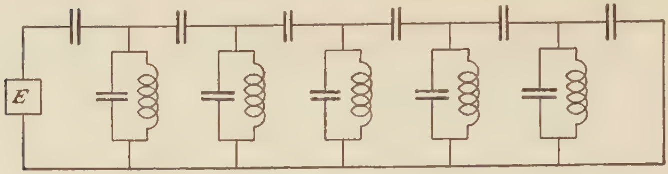

FIG. 10.

$$z_1 = \frac{1}{C_1 p}; z_2 = \frac{L_2}{1 + L_2 C_2 p^2}. \tag{52}$$

The frequencies are determined as before from the relation

$$\begin{aligned}\cos\frac{s\pi}{n} &= 1 + \frac{1}{2}\frac{z_1}{z_2} \\ &= 1 + \frac{1 + L_2 C_2 p^2}{2 L_2 C_1 p^2},\end{aligned} \tag{53}$$

which gives

$$\begin{aligned}p_s &= \frac{1}{\sqrt{L_2\left\{2C_1\left(\cos\frac{s\pi}{n} - 1\right) - C_2\right\}}} \\ &= \frac{1}{j\sqrt{L_2 C_2\left(1 + \frac{4C_1}{C_2}\sin^2\frac{s\pi}{2n}\right)}},\end{aligned} \tag{54}$$

and the frequencies of the system are

$$f_s = \frac{1}{2\pi\sqrt{L_2 C_2\left(1 + \frac{4C_1}{C_2}\sin^2\frac{s\pi}{2n}\right)}}. \tag{55}$$

The extreme frequencies at either end are given by

$$\left.\begin{aligned}f_1 &= \frac{1}{2\pi\sqrt{L_2 C_2\left(1 + \frac{4C_1}{C_2}\sin^2\frac{\pi}{2n}\right)}}, \\ f_n &= \frac{1}{2\pi\sqrt{L_2 C_2\left(1 + \frac{4C_1}{C_2}\right)}}.\end{aligned}\right\} \tag{56}$$

When n is very large, many sections, sin $s\pi/2n$ approaches zero value, and f_1 approaches the value $1/2\pi\sqrt{L_2C_2}$. By making

C_1/C_2 small, the difference between the extreme frequencies may be made as small as desired, and thus confine the frequencies of free vibrations of the system within a narrow range. This arrangement therefore offers another suitable method for band tuning, the width of the band determined by the difference between the two extreme frequencies f_1 and f_n.

Series Elements: Inductance and Capacity in Series; Shunt Elements: Inductance and Capacity in Parallel.—This arrangement gives

$$z_1 = L_1 p + \frac{1}{C_1 p}; \; z_2 = \frac{L_2 p}{1 + L_2 C_2 p^2}; \qquad (57)$$

and

$$\cos\frac{s\pi}{n} = 1 + \frac{1}{2}\frac{z_1}{z_2} = 1 + \frac{\left(L_1 p + \dfrac{1}{C_1 p}\right)\left(1 + L_2 C_2 p^2\right)}{2L_2 p} \qquad (58)$$

This transforms to the following:

$$-4L_2 p \sin^2\frac{s\pi}{2n} = L_1 p + L_1 L_2 C_2 p^3 + \frac{1}{C_1 p} + \frac{L_2 C_2}{C_1} p,$$

or

$$L_1 L_2 C_1 C_2 p^4 + \left(L_2 C_2 + L_1 C_1 + 4L_2 C_1 \sin^2\frac{s\pi}{2n}\right)p^2 + 1 = 0.$$

Solving for p_s^2 gives

$$p_s^2 = \frac{-\left(L_2 C_2 + L_1 C_1 + 4L_2 C_1 \sin^2\dfrac{s\pi}{2n}\right) \pm \sqrt{\left(L_2 C_2 + L_1 C_1 + 4L_2 C_1 \sin^2\dfrac{s\pi}{2n}\right)^2 - 4L_1 L_2 C_1 C_2}}{2L_1 L_2 C_2 C_2}, \qquad (59)$$

and the frequencies are given by

$$f_s = \frac{1}{2\pi}\sqrt{\frac{\left(L_2 C_2 + L_1 C_1 + 4L_2 C_1 \sin^2\dfrac{s\pi}{2n}\right) \pm \sqrt{\left(L_2 C_2 + L_1 C_1 + 4L_2 C_1 \sin^2\dfrac{s\pi}{2n}\right)^2 - 4L_1 L_2 C_1 C_2}}{2L_1 L_2 C_1 C_2}}. \qquad (60)$$

If we make n very large, then for $s = 1$, $\sin^2 \frac{s\pi}{2n} = 0$, and

$$\left.\begin{aligned} f_1 &= \frac{\frac{1}{2\pi}\sqrt{(L_2C_2 + L_1C_1) \pm \sqrt{(L_2C_2 - L_1C_1)^2}}}{\sqrt{2L_1L_2C_1C_2}} \\ \text{For } s = n \qquad & \\ f_n &= \frac{1}{2\pi}\sqrt{\frac{(L_2C_2 + L_1C_1 + 4L_2C_1) \pm \sqrt{(L_2C_2 + L_1C_1 + 4L_2C_1)^2 - 4L_1L_2C_1C_2}}{2L_1L_2C_1C_2}} \end{aligned}\right\} \quad (61)$$

The free periods of vibrations of the system have two frequency ranges, one between the values of f_1 and f_n, when the positive signs are used in the double-sign terms of (60) and (61), and the other when the negative signs are used. When $L_1C_1 = L_2C_2$, the frequencies are given by

$$\left.\begin{aligned} f_s &= \frac{1}{2\pi}\sqrt{\frac{2 + 4\sin^2\frac{s\pi}{2n} \pm \sqrt{\left(2 + 4\sin^2\frac{s\pi}{2n}\right)^2 - 4}}{2LC}} \\ &= \frac{1}{2\pi}\sqrt{\frac{1 + 2\sin^2\frac{s\pi}{2n} \pm 2\sin\frac{s\pi}{2n}\sqrt{\left(1 + \sin^2\frac{s\pi}{2n}\right)}}{LC}}. \end{aligned}\right\} \quad (62)$$

Magnetically Coupled Filter-circuit System.—The various types of filter-circuit systems considered so far are all of the

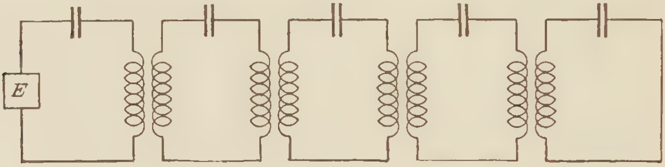

FIG. 11.

direct-coupled types, the recurrent sections connected directly in tandem. There is no need, however, to impose any limitation on the type of couplings to be employed between the recurrent sections. Results, similar to those obtained for the direct-coupled filter circuit, are obtainable with magnetically coupled circuits or mixed coupling. We shall investigate now the properties and characteristics of a magnetically coupled filter-circuit system, that is, one consisting of a number of successively magnetically coupled circuits, as shown diagrammatically in Fig. 11.

If we designate by z the impedance of each circuit, and by M the mutual inductance between any two adjacent circuits, we have the following system of equations for the current distribution in the circuits:

$$\left.\begin{array}{r} zi_1 + Mpi_2 = E, \\ zi_2 + Mpi_1 + Mpi_3 = 0, \\ zi_3 + Mpi_2 + Mpi_4 = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ zi_m + Mpi_{m-1} + Mpi_{m+1} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ zi_n + Mpi_{n-1} = 0. \end{array}\right\} \tag{63}$$

Assume a solution of the following form:

$$i_m = A\epsilon^{m\gamma} + B\epsilon^{-m\gamma}. \tag{64}$$

Substituting this in any equation of (63) except the first and last gives

$$z(A\epsilon^{m\gamma} + B\epsilon^{-m\gamma}) + Mp(A\epsilon^{(m-1)\gamma} + B\epsilon^{-(m-1)\gamma}) + Mp(A\epsilon^{(m+1)\gamma} + B\epsilon^{-(m+1)\gamma}) = 0,$$

and rearranging,

$$(z + Mp\epsilon^{\gamma} + Mp\epsilon^{-\gamma})(A\epsilon^{m\gamma} + B\epsilon^{-m\gamma}) = 0. \tag{65}$$

The solution given by (64) will satisfy the equations of (63), except the first and last, if we assign a value to γ to satisfy the relation.

$$z + Mp(\epsilon^{\gamma} + \epsilon^{-\gamma}) = 0,$$

or

$$\cosh\gamma = \frac{-z}{2Mp}. \tag{66}$$

For the first and last equations we have

$$\left.\begin{array}{r} z(A\epsilon^{\gamma} + B\epsilon^{-\gamma}) + Mp(A\epsilon^{2\gamma} + B\epsilon^{-2\gamma}) = E, \\ z(A\epsilon^{n\gamma} + B\epsilon^{-n\gamma}) + Mp(A\epsilon^{(n-1)\gamma} + B\epsilon^{-(n-1)\gamma}) = 0. \end{array}\right\} \tag{67}$$

Two equations from which the constants A and B are determined, thus:

$$A = \frac{E\epsilon^{-n\gamma}(z + Mp\epsilon^{\gamma})}{\epsilon^{-(n-1)\gamma}(z + Mp\epsilon^{\gamma})^2 - \epsilon^{(n-1)\gamma}(z + Mp\epsilon^{-\gamma})^2},$$

$$B = \frac{-E\epsilon^{n\gamma}(z + Mp\epsilon^{-\gamma})}{\epsilon^{-(n-1)\gamma}(z + Mp\epsilon^{\gamma})^2 - \epsilon^{(n-1)\gamma}(z + Mp\epsilon^{-\gamma})^2}.$$

Taking into account the relation given by (66), the above reduce to

$$\left.\begin{aligned} A &= \frac{-E\epsilon^{-(n+1)\gamma}}{Mp(\epsilon^{-(n+1)\gamma} - \epsilon^{(n+1)\gamma})}, \\ B &= \frac{E\epsilon^{(n+1)\gamma}}{Mp(\epsilon^{-(n+1)\gamma} - \epsilon^{(n+1)\gamma})}. \end{aligned}\right\} \tag{68}$$

Introducing these values in (64), we obtain the expression for the current in any of the circuits, say the mth circuit,

$$\dot{i}_m = \frac{E\{\epsilon^{(m-n-1)\gamma} - \epsilon^{-(m-n-1)\gamma}\}}{Mp\{\epsilon^{(n+1)\gamma} - \epsilon^{(n+1)\gamma}\}}. \tag{69}$$

Expressed in hyperbolic functions

$$\dot{i}_m = \frac{E \sinh (m - n - 1)\gamma}{Mp \sinh (n + 1)\gamma}. \tag{70}$$

For the last, the nth circuit,

$$\dot{i}_n = \frac{-E \sinh \gamma}{Mp \sinh (n + 1)\gamma}. \tag{71}$$

The complete solution is readily developed from the above expression by the application of the expansion theorem. The determinantal equation is

$$Z(p) = Mp \sinh (n + 1)\gamma = 0. \tag{72}$$

Hence,

$$\left.\begin{aligned} (n + 1)\gamma &= js\pi, \\ \gamma &= \frac{js\pi}{n + 1};\ s = 0, 1, 2, 3, \cdots n + 1. \end{aligned}\right\} \tag{73}$$

The values $s = 0$ and $s = n + 1$ are to be disregarded, because for these values, the numerator in (71) is also zero. We take, therefore, only the values of s from 1 to n.

The values of p corresponding to the roots of equation (72) are obtained from the relation given by (66). If each circuit consists of an inductance and capacity in series, $z = Lp + R + 1/Cp$, and by (66)

$$\cos \frac{s\pi}{n + 1} = \frac{-\left(Lp + R + \dfrac{1}{Cp}\right)}{2Mp},$$

which gives a second-degree equation in p,

$$\left\{2LC + MC \cos \frac{s\pi}{n+1}\right\}p^2 + 2RCp + 1 = 0.$$

Solving, we obtain the values of p corresponding to the roots of equation (72) as follows:

$$p_s = \alpha_s + j\beta_s, \tag{74}$$

where

$$\alpha_s = \frac{-R}{2\left(L + 2M \cos \frac{s\pi}{n+1}\right)},$$

$$\beta_s = \frac{\sqrt{LC + 2MC \cos \frac{s\pi}{n+1} - \frac{R^2C^2}{4L^2}}}{LC + 2MC \cos \frac{s\pi}{n+1}}. \tag{75}$$

We also have

$$\frac{\partial Z(p)}{\partial p} = Mp \cosh (n+1)\gamma \frac{\partial (n+1)\gamma}{\partial p}. \tag{76}$$

By (66),

$$\sinh \gamma \frac{\partial \gamma}{\partial p} = \frac{1}{2Mp^2}\left(R + \frac{2}{Cp}\right).$$

Substituting this value of $\partial\gamma/\partial p$ in (76), gives

$$\frac{\partial Z(p)}{\partial p} = \frac{(n+1)}{2p}\left(R + \frac{2}{Cp}\right)\frac{\cos (n+1)\gamma}{\sinh \gamma}. \tag{77}$$

Introducing the values of p_s and $\partial Z(p)/\partial p$ from (74) and (77) in the expansion formula, and remembering that it is to be multiplied by $\sinh \gamma$, the numerator of (71), we finally obtain the completely developed solution for the current in the nth circuit.

$$i_n = E\sum \frac{2 \sinh^2 \gamma \epsilon^{(+\alpha_s + j\beta_s)t}}{(n+1)\left(R + \frac{2}{C(\alpha_s + j\beta_s)}\right) \cosh (n+1)\gamma}$$

$$= 2E\sum \frac{\sin^2 \frac{s\pi}{n+1} \epsilon^{(+\alpha_s \pm j\beta_s)t}}{(n+1)\left(R + \frac{2}{C(\alpha_s + j\beta_s)}\right) \cos (s\pi)}. \tag{78}$$

Including both terms in each double-sign term, the above reduces by simple algebraic transformation to the following:

$$i_n = 4E\sum \frac{\epsilon^{\alpha_s t}\left\{\left(R + \dfrac{2\alpha_s}{C(\alpha_s^2+\beta_s^2)}\right)\cos\beta_s t - \dfrac{2\beta_s}{C(\alpha_s^2+\beta_s^2)}\sin\beta_s t\right\}\sin^2\dfrac{s\pi}{n+1}}{(n+1)\left\{R^2 + \dfrac{4R\alpha_s}{C(\alpha_s^2+\beta_s^2)} + \dfrac{4}{C^2(\alpha_s^2+\beta_s^2)}\right\}\cos(s\pi)}. \tag{79}$$

By (75), however, we have

$$\alpha_s^2 + \beta_s^2 = \frac{1}{C\left(L + 2M\cos\dfrac{s\pi}{n+1}\right)},$$

$$\frac{\alpha_s}{\alpha_s^2+\beta_s^2} = \frac{-RC}{2}.$$

Making these substitutions, (79) simplifies to

$$i_n = 4E\sum\frac{\epsilon^{\alpha_s t}\left\{2R\cos\beta_s t - 2\beta_s\left(L+2M\cos\dfrac{s\pi}{n+1}\right)\sin\beta_s t\right\}\sin^2\dfrac{s\pi}{n+1}}{(n+1)\left\{3R^2 + \dfrac{4\left(L+2M\cos\dfrac{s\pi}{n+1}\right)}{C}\right\}\cos(s\pi)}. \tag{80}$$

This is the complete solution for the current in the nth circuit of n successively coupled circuits. The current consists of n separate oscillations, each of an independent frequency, damping, and amplitude. The frequencies and damping factors are given by (75). Thus:

For two circuits, $n = 2$.

$$\alpha_1 = \frac{-R}{2L(1+K)}; \qquad \alpha_2 = \frac{-R}{2L(1-K)}.$$

$$f_1 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1+K)}}; \; f_2 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1-K)}}.$$

$$K = \frac{M}{L} \tag{81}$$

For three circuits, $n = 3$.

$$\alpha_1 = \frac{-R}{2L(1 + 1.414K)}; \qquad \alpha_2 = \frac{-R}{2L}; \qquad \alpha_3 = \frac{-R}{2L(1 - 1.414K)}.$$

$$f_1 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1+1.414K)}}; f_2 = \frac{1}{2\pi\sqrt{LC}}; f_3 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1-1.414K)}}.$$

For four circuits, $n = 4$.

$$\alpha_1 = \frac{-R}{2L(1 + 1.616K)}; \qquad \alpha_2 = \frac{-R}{2L(1 + 0.616K)};$$

$$\alpha_3 = \frac{-R}{2L(1 - 0.616K)}; \qquad \alpha_4 = \frac{-R}{2L(1 - 1.616K)}.$$

$$f_1 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1 + 1.616K)}}; f_2 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1 + 0.616K)}};$$

$$f_3 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1 - 0.616K)}}; f_4 = \frac{1}{2\pi}\frac{1}{\sqrt{LC(1 - 1.616K)}}.$$

If we neglect the resistances of the circuits, equation (80) simplifies to the following:

$$i_n = \frac{2EC}{n+1}\sum_{s=1}^{s=n}\frac{\beta_s \sin \beta_s t}{\cos (s\pi)} \sin^2 \frac{s\pi}{n+1}. \tag{82}$$

For two-coupled circuits, $n = 2$, the current in the secondary circuit is given by

$$i_2 = \frac{2EC}{3}\left\{-\beta_1 \sin \beta_1 t \sin^2\left(\frac{\pi}{3}\right) + \beta_2 \sin \beta_2 t \sin^2\left(\frac{2\pi}{3}\right)\right\}$$

$$= \frac{EC}{2}\{\beta_2 \sin \beta_2 t - \beta_1 \sin \beta_1 t\}\cdot \tag{83}$$

which is in exact accord with (62) of Chap. II, the formula arrived at for this condition by another method.

The corresponding dampings of the two oscillations of the two-couple circuits are

$$\alpha_1 = \frac{-R}{2L(1 + K)}; \alpha_2 = \frac{-R}{2L(1 - K)}.$$

For a three-coupled circuit system, $n = 3$, the current in the third circuit is given by

$$i_3 = \frac{EC}{2}\left\{\frac{1}{2}\beta_1 \sin \beta_1 t - \beta_2 \sin \beta_2 t + \frac{1}{2}\beta_1 \sin \beta_1 t\right\}\cdot \tag{84}$$

The corresponding dampings are

$$\alpha_1 = \frac{-R}{2L(1 + 1.414K)};\ \alpha_2 = \frac{-R}{2L};\ \alpha_3 = \frac{-R}{2L(1 - 1.414K)}.$$

For a four-coupled circuit system, $n = 4$, the current in the fourth circuit is given by

$$i_4 = 0.4EC \left\{ \begin{array}{l} -0.3455\beta_1 \sin \beta_1 t + 0.9046\beta_2 \sin \beta_2 t \\ -0.9046\beta_3 \sin \beta_3 t + 0.3455\beta_4 \sin \beta_4 t \end{array} \right\}. \tag{85}$$

and the corresponding dampings,

$$\alpha_1 = \frac{-R}{2L(1 + 1.616K)};\ \alpha_2 = \frac{-R}{2L(1 + 0.616K)};$$
$$\alpha_3 = \frac{-R}{2L(1 - 0.616K)};\ \alpha_4 = \frac{-R}{2L(1 - 1.616K)}.$$

The method followed here made it possible to obtain the complete solution for the current distribution in any number of coupled circuits; to determine the frequencies, dampings, and amplitudes of the several oscillatory components. The only limitation imposed is that the electrical constants of the individual circuits L, C, and R, and the mutual inductance between any two circuits are the same for all the circuits.

Resonance Effects of Multiperiodic Circuit System.—We have established, now, that a circuit system consisting of a number of coupled circuits has a number of periods of free oscillations, each oscillation characterized by an independent frequency, damping, and amplitude. This is perfectly general, applying to all types of couplings, magnetic, static, or direct. We may reasonably infer from this that if we should impress alternating voltages on a circuit system of this kind, it would respond resonantly to several frequencies, which are precisely the frequencies of the free oscillations of the system, and to those frequencies only. Hence, a resonance curve of a circuit system of this kind would, in effect, be the resultant of several resonance curves, each corresponding to one of the frequencies of the free oscillations of the system. If the circuit system is designed for small frequency separation between adjacent frequencies, then, obviously, parts of adjacent resonance curves would overlap, the curves would partly merge, and the effect would be as if the system were in partial resonance even for the frequencies which are between resonance frequencies. If the circuit system is designed to have a large number of

resonance frequencies, all confined within a narrow range of frequencies, the resonance curves would be crowded together and the effect approximate a uniform resonance for all the frequencies within that range of frequencies. This is sometimes designated as *band tuning.*

The author has emphasized this point, at the expense, perhaps, of repetition, because of the confusion in the literature of this subject regarding the meaning of so-called *band tuning.* Further to bring out this point, I have worked out a numerical problem to show the resonance effects for impressed voltages of different frequencies in a two, three, and four magnetically coupled circuit, and plotted the resonance curves which are given in Figs. 12, 13, and 14.

To compute these curves, use is made of the basic formula

$$\dot{i}_n = \frac{E \sinh \gamma}{Mp \sinh (n + 1)\gamma}.$$

For impressed alternating voltages, this formula, with the change of p to $j\omega$, gives the steady-state current. We also have the auxiliary formula

$$\cosh = \frac{-z}{2Mp},$$

and for $p = j\omega$,

$$\cosh \gamma = \frac{-\left(L\omega - \dfrac{1}{C\omega}\right)}{2M\omega},$$

neglecting resistances.

It is clear that, since the frequencies of the free oscillations were determined from the determinant equation $\sinh (n + 1)\gamma = 0$, then obviously reversing the process, impressing voltages of these frequencies on the system would also give $\sinh (n + 1)\gamma = 0$, and the currents for these frequencies would be infinitely large if the resistances are neglected. This can be readily verified working backward. Substituting the expression for β, defining the frequencies of the free oscillations, in place of ω, we have

$$\begin{aligned} \sinh \gamma &= \sqrt{\cosh^2 \gamma - 1} \\ &= \sqrt{\frac{\left(L\omega - \dfrac{1}{C\omega}\right)^2}{4M^2\omega^2} - 1}. \end{aligned} \qquad (86)$$

Substituting for ω the free oscillation frequency value

$$\omega = \frac{1}{\sqrt{LC\left(1 + \frac{2M}{L}\cos\frac{s\pi}{n+1}\right)}}, \tag{87}$$

the above transforms to

$$\sinh\gamma = \frac{\sqrt{-\frac{L}{C\left(1+\frac{2M}{L}\cos\frac{s\pi}{n+1}\right)} + \frac{L}{C}\left(1+\frac{2M}{L}\cos\frac{s\pi}{n+1}\right) - \frac{4M^2}{LC\left(1+\cos\frac{s\pi}{n+1}\right)}}}{\frac{2M}{\sqrt{LC\left(1+\frac{2M}{L}\cos\frac{s\pi}{n+1}\right)}}}. \tag{88}$$

which reduces to the following:

$$\sinh\gamma = \sqrt{-1 + \cos^2\frac{s\pi}{n+1}} = j\sin\frac{s\pi}{n+1}.$$

Hence,

$$\gamma = j\frac{s\pi}{n+1},$$

and

$$\sinh\,(n+1)\gamma = \sin\,(s\pi) = 0. \tag{89}$$

For these frequencies, the current in the circuit is infinite for zero resistances. To determine the general form of the current curve in response to voltages of different frequencies, it will be sufficient to calculate the current values for frequencies in between resonance frequencies and for frequencies on either side of the two extreme resonance frequencies. The following values of the constants of the circuits were assumed:

$$L = 0.5\ mh;\ C = 0.0002\ mf;\ M = 0.05\ mh.$$

A two-coupled circuit system:

The resonance frequencies are $1/2\pi\ 10^7/\sqrt{11}$, and $1/2\pi\ 10^7/\sqrt{9}$. For the frequency in between these frequencies, say $1/2\pi\ 10^7/\sqrt{10}$, $\cosh\omega = 0$, and $\sinh\gamma = j$; $\gamma = j\pi/4$ and $\sinh\,(n+1)\gamma = \sinh 3j\pi/4 = j$

For $f_4 = \frac{10^7}{2\pi\sqrt{14}}$; $\sinh\gamma = 1.732$ and $\sinh 3\gamma = 25.96$.

For $f_5 = \frac{10^7}{2\pi\sqrt{6}}$; $\sinh\gamma = 1.732$ and $\sinh 3\gamma = 25.96$.

A three-coupled circuit system:
The resonance frequencies are

$$f_1 = \frac{1}{2\pi} \frac{10^7}{\sqrt{11.414}}; f_2 = \frac{10^7}{2\pi\sqrt{10}}; f_3 = \frac{10^7}{2\pi\sqrt{8.586}}.$$

For $f_4 = \frac{10^7}{2\pi \times 3.27}$; sinh $\gamma = j0.937$ and sinh $4\gamma = j0.49$.

$f_5 = \frac{10^7}{2\pi \times 3.05}$; sinh $\gamma = j0.937$ and sinh $4\gamma = j0.49$.

$f_6 = \frac{10^7}{2\pi \times 3.5}$; sinh $\gamma = 0.515$ and sinh $4\gamma = 2.094$.

$f_7 = \frac{10^7}{2\pi \times 2.75}$; sinh $\gamma = 0.698$ and sinh $4\gamma = 3.454$.

FIG. 12.

A four-coupled circuit system:
The resonance frequencies are:

$$f_1 = \frac{10^7}{2\pi\sqrt{11.618}}; f_2 = \frac{10^7}{2\pi\sqrt{10.618}}; f_3 = \frac{10^7}{2\pi\sqrt{9.382}}; f_4 = \frac{10^7}{2\pi\sqrt{8.382}}.$$

For $f_5 = \frac{0.3 \times 10^7}{2\pi}$; sinh $\gamma = j0.83$; sinh $5\gamma = -j0.983$

For $f_6 = \frac{0.316 \times 10^7}{2\pi}$; sinh $\gamma = j1.0$; sinh $5\gamma = -j1.00$

For $f_7 = \dfrac{0.334 \times 10^7}{2\pi}$; $\sinh \gamma = j0.84$; $\sinh 5\gamma = -j0.983$

For $f_8 = \dfrac{0.28 \times 10^7}{2\pi}$; $\sinh \gamma = 0.95$; $\sinh 5\gamma = 34.00$.

For $f_9 = \dfrac{0.36 \times 10^7}{2\gamma}$; $\sinh \gamma = 0.55$; $\sinh 5\gamma = 6.9$.

From these values, the graphs of Figs. 12, 13, and 14, are plotted

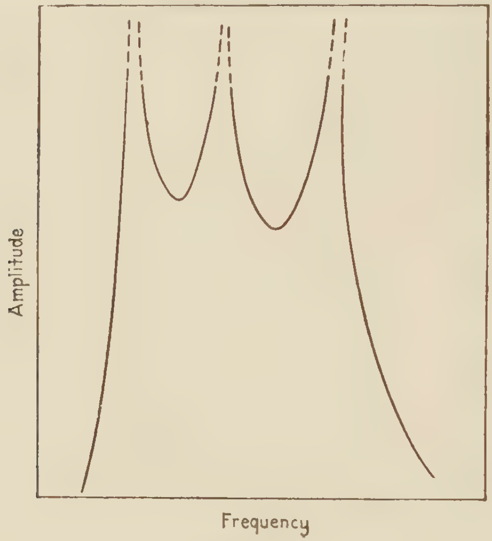

FIG. 13.

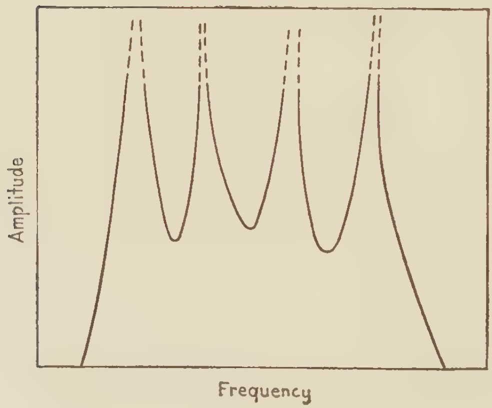

FIG. 14.

CHAPTER IV

OCEAN CABLES

CIRCUITS OF DISTRIBUTED CAPACITY AND RESISTANCE

A considerable portion of the second volume of Heaviside's "Electromagnetic Theory" is devoted to a discussion of problems relating to wave propagation on cables. A large number of problems touching various phases of the subject are analyzed, in which direct operational solutions and the expansion theorem are applied. It is altogether beyond the scope of this book to give as comprehensive a discussion here; nevertheless, the topics selected are such as to cover the most important phases of the subject and at the same time show the application of the operational solutions in a variety of forms to serve as a guide in the solution of practical problems that are apt to arise in connection with this subject.

For a cable of negligible inductance and leakage, the relations between the current and voltage at any point on the cable, distance x from the transmission end, are given by the following equations:

$$\left.\begin{aligned} Ri &= -\frac{dV}{dx}, \\ CpV &= -\frac{di}{dx}. \end{aligned}\right\} \tag{1}$$

R and C are the resistance and capacity per unit length of cable, and $p = d/dt$.

From these equations, we readily derive the propagation equations for either the voltage or the current, as follows:

$$\left.\begin{aligned} RCpV &= \frac{d^2V}{dx^2}, \\ RCpi &= \frac{d^2i}{dx^2}. \end{aligned}\right\} \tag{2}$$

It is sufficient if a solution is obtained for either one of the equation (2); the other is readily evaluated by the aid of the

relations (1). Consider the voltage equation and assume a solution,

$$V = A\epsilon^{Kx} + B\epsilon^{-Kx}. \tag{3}$$

This satisfies the differential equation if

$$K = \sqrt{RCp}. \tag{4}$$

A and B are independent of x, but functions of time, and depend on the terminal conditions. By either of equations (1), we get the following expression for the current:

$$i = \frac{K}{R}\{-A\epsilon^{Kx} + B\epsilon^{-Kx}\}. \tag{5}$$

These are general solutions involving indeterminate time functions A and B, which must be separately evaluated in each special case.

CABLE OF INFINITE LENGTH

In the case of a uniform cable of infinite length, any electric impulse impressed on the cable travels along the cable without suffering any reflection at any point on the cable, and it is because of the absence of any reflections that the mathematical solution of the problem is considerably simplified. We shall, therefore, begin the investigation of this subject by considering some problems relating to cables of infinite length.

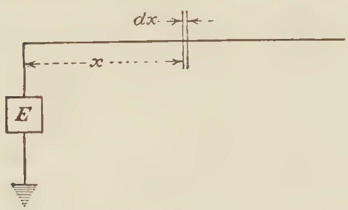

FIG. 15.

Assume a steady voltage E impressed at the transmitting end of the cable $x = 0$, to determine the voltage and current at any point on the cable distance x from the transmitting end. The terminal conditions for this case are

$$x = 0;\ V = E,$$
$$x = \infty;\ V = 0,$$

which give

$$A = 0 \text{ and } B = E.$$

Hence,

$$\left.\begin{aligned} V &= E\epsilon^{-Kx}, \\ i &= \frac{K}{R}E\epsilon^{-Kx}. \end{aligned}\right\} \tag{6}$$

At the transmission end, $x = 0$

$$\left.\begin{aligned} V_0 &= E, \\ i_0 &= E\sqrt{\frac{Cp}{R}}. \end{aligned}\right\} \tag{7}$$

What meaning are we to attach to the fractional derivative $\sqrt{p} = \sqrt{d/dt}$ which appears in above equations? Heaviside asserts, and apparently he discovered it experimentally, that if we operate by $\sqrt{p}$ on unity function, that is, a function which has zero value for $t < 0$ and is constant for all values of $t > 0$, the result is $1/\sqrt{\pi t}$. That is,

$$\sqrt{p}1 = \frac{1}{\sqrt{\pi t}}. \tag{8}$$

There is no question about the correctness of this relation; many problems worked out by Heaviside on this assumption yield correct results. Farther on (see pages 73, 97), we shall give partial proof establishing the validity of this assumption. With the definition of $\sqrt{p}1$ as given by (8), the current at the transmitting end is given by

$$i_0 = E\sqrt{\frac{C}{\pi R t}}. \tag{9}$$

Obviously, operating by $\sqrt{p}$ on any constant quantity gives $1/\sqrt{\pi t}$ times that constant. Hereafter, if not otherwise specified, unity function is to be understood.

To obtain expressions for the voltage and current at any point on the cable, distance x from the transmitting end, we must algebrize equations (6) which involve operating by ϵ^{-Kx}. Expand ϵ^{-Kx} in a series, and

$$\begin{aligned} V &= E\left\{1 - Kx + \frac{K^2x^2}{2!} - \frac{K^3x^3}{3!} + \frac{K^4x^4}{4!} - \cdots\right\} \\ &= E\left\{1 - (RCp)^{1/2}x + \frac{(RCp)x^2}{2!} - \frac{(RCp)^{3/2}x^3}{3!} + \cdots\right\}. \end{aligned} \tag{10}$$

The terms of integral powers of p in the above expression are to be disregarded, because they imply complete differentiation of a constant, in this case unity, which gives, of course, zero. Hence, (10) reduces to

$$V = E\left\{1 - (RCp)^{1/2}x - \frac{(RCp)^{3/2}x^3}{3!} - \frac{(RCp)^{5/2}x^5}{5!} - \cdots\right\}. \tag{11}$$

Since

$$\sqrt{p} = \frac{1}{\sqrt{\pi t}},$$

we get by successive differentiation

$$\left.\begin{aligned} p^{3/2} &= p\sqrt{p} = -\frac{1}{2\sqrt{\pi}}t^{-3/2} \\ p^{5/2} &= p^2\sqrt{p} = \frac{3}{2\cdot 2\sqrt{\pi}}t^{-5/2} \\ p^{7/2} &= p^3\sqrt{p} = -\frac{3\cdot 5}{2\cdot 2\cdot 2\sqrt{\pi}}t^{-7/2} \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{aligned}\right\} \tag{12}$$

Introducing these values in (11) gives

$$V = E\left\{1 - \frac{1}{\sqrt{\pi}}\left(\frac{RC}{t}\right)^{1/2}x + \frac{1}{2\cdot 3!\sqrt{\pi}}\left(\frac{RC}{t}\right)^{3/2}x^3 - \frac{3}{2\cdot 2\cdot 5!\sqrt{\pi}}\left(\frac{RC}{t}\right)^{5/2}x^5 + \cdots\right\}$$

or

$$\frac{V}{E} = 1 - \left(\frac{RCx^2}{\pi t}\right)^{1/2}\left\{1 - \frac{1}{3}\left(\frac{RCx^2}{4t}\right) + \frac{1}{5\cdot 2!}\left(\frac{RCx^2}{4t}\right)^2 - \cdots\right\}, \tag{13}$$

which is the expression for the voltage at any point on the cable. It is seen that it is a function of $RCx^2/4t$, and if we put this for brevity y^2, we have

$$\frac{V}{E} = 1 - \frac{2}{\sqrt{\pi}}\left\{y - \frac{y^3}{3} + \frac{y^5}{5.2!} - \frac{y^7}{7.3!} + \cdots\right\}$$

which may be put in the form of a definite integral.

$$\frac{V}{E} = 1 - \frac{2}{\sqrt{\pi}}\int_0^y \epsilon^{-y^2}dy. \tag{14}$$

This integral term is sometimes designated as the *error function* and denoted by erf. y for which tables have been worked out. A short table is given in the appendix.

The expression for the current in the cable could be obtained in the same way, utilizing the same operations as in the derivation

of the voltage formula (13). It is simpler, however, to derive the expression for the current from (13) by the circuital relations (1).

$$i = -\frac{1}{R}\frac{dV}{dx}$$

$$= \frac{E}{R}\left\{\left(\frac{RC}{\pi t}\right)^{\frac{1}{2}} - \frac{1}{2\cdot 2\sqrt{\pi}}\left(\frac{RC}{t}\right)^{\frac{3}{2}} x^2 + \frac{3\cdot 5}{2\cdot 2\cdot 5!}\left(\frac{RC}{t}\right)^{\frac{5}{2}} x^4 - \cdots\right\}$$

$$= \frac{E}{R}\left(\frac{RC}{\pi t}\right)^{\frac{1}{2}}\left\{1 - \frac{RC}{4t}x^2 + \frac{1}{2!}\left(\frac{RC}{4t}\right)^2 x^4 - \cdots\right\}$$

$$= \frac{E\sqrt{C}}{\sqrt{R\pi t}}\epsilon^{-\frac{RC}{4t}x^2}. \tag{15}$$

Equations (13) or (14) and (15) determine the voltage and current at any point on the cable for any time t.

CABLE OF FINITE LENGTH

In a cable of finite length, the character of the voltage and current depends not only on the electrical characteristics of the cable itself but also on the terminal conditions at the ends of the cable, the latter determining the character of the reflections at either end. The voltage and current waves on the cable are, in fact, composite waves, the resultant of practically an infinite series of waves arising from the repeated reflections at either end.

We shall consider first the simplest cases, those of a free cable, the ends either grounded or open, no terminal impedances, in which case the reflected waves are of the same character as the incident waves with or without reversal of signs. For a grounded end, the voltage wave is reflected negatively; for an open end, the voltage wave is reflected positively.

Whatever the terminal conditions are, the equations for the voltage and current in the cable are still given by (3) and (5)

$$\left.\begin{aligned} V &= A\epsilon^{Kx} + B\epsilon^{-Kx}, \\ i &= \sqrt{\frac{Cp}{R}}(-A\epsilon^{Kx} + B\epsilon^{-Kx}). \end{aligned}\right\} \tag{16}$$

The factors A and B are time functions, but independent of x, and are determinable from the terminal conditions.

A Cable of Length l Grounded at Both Ends.—A constant voltage E applied at the end $x = 0$. We have for this case,

$$x = 0;\ V = E,$$

$$x = l;\ V = 0.$$

which gives on substitution in (16),

$$A + B = E,$$
$$A\epsilon^{Kl} + B\epsilon^{-Kl} = 0,$$

from which A and B are determined as follows:

$$\left.\begin{aligned} A &= \frac{-E\epsilon^{-Kl}}{\epsilon^{Kl} - \epsilon^{-Kl}}, \\ B &= \frac{E\epsilon^{Kl}}{\epsilon^{Kl} - \epsilon^{-Kl}}. \end{aligned}\right\} \tag{17}$$

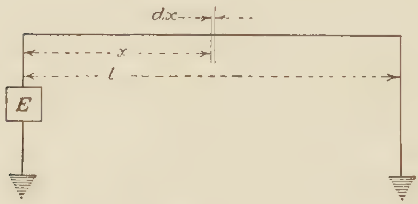

FIG. 16.

Introducing these values in (16) gives

$$\left.\begin{aligned} V &= \frac{E\{\epsilon^{K(l-x)} - \epsilon^{-K(l-x)}\}}{\epsilon^{Kl} - \epsilon^{-Kl}}, \\ i &= \frac{E\sqrt{\dfrac{Cp}{R}}\{\epsilon^{K(l-x)} + \epsilon^{-K(l-x)}\}}{\epsilon^{Kl} - \epsilon^{-Kl}}. \end{aligned}\right\} \tag{18}$$

These equations may be put in this form:

$$\left.\begin{aligned} V &= \frac{E \sinh K(l-x)}{\sinh Kl}, \\ i &= E\sqrt{\frac{Cp}{R}}\frac{\cosh K(l-x)}{\sinh Kl}. \end{aligned}\right\} \tag{19}$$

To develop the solution from the above equations, we may utilize either the direct operational process or apply the expansion theorem. We shall adopt the latter as being the simpler method. Heaviside carried through in detail the operational process.

The expansion-theorem formula, of which the derivation is given in Chap. II, is

$$i \text{ or } V = \frac{E}{Z(p)_{p=0}} + E\sum_{n=1}^{n=m} \frac{\epsilon^{p_n t}}{p_n \dfrac{\partial Z(p)}{\partial p}}\, p = p_n, \tag{20}$$

which may be applied for the determination of either V or i. For the determination of V, we note that the determinantal equation is

$$Z(p) = \frac{\sinh Kl}{\sinh K(l - x)} = 0 \tag{21}$$

Hence,

$$Kl = jn\pi; \; n = 1, 2, 3, 4 \cdots$$

and

$$K = \sqrt{RCp} = j\frac{n\pi}{l},$$

from which we get

$$p = -\frac{n^2\pi^2}{RCl^2}. \tag{22}$$

There are an infinite number of values of p given by equation (22) which satisfy the determinantal equation. We also have

$$\frac{\partial Z(p)}{\partial p} = \cosh Kl \frac{\partial(Kl)}{\partial p},$$

but

$$\frac{\partial(Kl)}{\partial p} = \frac{l}{2}\sqrt{\frac{RC}{p}} = \frac{RCl}{2\sqrt{RCp}} = \frac{RCl^2}{2jn\pi},$$

therefore,

$$\frac{\partial Z(p)}{\partial p} = \frac{RCl^2}{2jn\pi} \cosh Kl = \frac{RCl^2}{2jn\pi} \cos n\pi. \tag{23}$$

For $p = 0$, we note that $\sinh Kl = 0$, but so is also the numerator in (19), $\sinh K(l - x) = 0$, which would make the expression indeterminate. By taking, however, the value of $\sinh K(l - x) / \sinh Kl$ as p approaches zero, we get

$$\frac{\sinh K(l - x)}{\sinh Kl} = \frac{K(l - x)}{Kl} = 1 - \frac{x}{l},$$

and

$$Z(p)_{p=0} = 1 - \frac{x}{l}. \tag{24}$$

Also,

$$\sinh K(l - x) = j \sin \frac{n\pi}{l}(l - x) = j \sin n\pi\left(1 - \frac{x}{l}\right)$$

$$= -j \cos n\pi \sin \frac{n\pi x}{l}.$$

Introducing these values in (20) we obtain

$$V = E\left(1 - \frac{x}{l}\right) + E\sum_{n-1}^{n=\infty}\frac{\epsilon^{-\frac{n^2\pi^2}{RCl^2}t} j \cos n\pi \sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{RCl^2} RCl^2 \frac{\cos n\pi}{2jn\pi}}$$

$$= E\left(1 - \frac{x}{l}\right) - \frac{2E}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{n}\sin\frac{n\pi x}{l}\,\epsilon^{-\frac{n^2\pi^2}{RCl^2}t}, \qquad (25)$$

which is the complete solution for the voltage at any point on the cable for any time t after the application of the voltage. For infinite time, we get the steady-state condition which is, by (25), for $t = \infty$,

$$V = E\left(1 - \frac{x}{l}\right). \qquad (26)$$

The expression for the current may be derived in the same way by the application of the expansion theorem to the second equation of (19), or we may obtain it from (25) by the circuital relation

$$Ri = -\frac{dY}{dx},$$

which gives

$$i = \frac{E}{Rl} + \frac{2E}{Rl}\sum_{n=1}^{n=\infty}\cos\frac{n\pi x}{l}\,\epsilon^{-\frac{n^2\pi^2}{RCl^2}t}. \qquad (27)$$

If l is made infinitely long in above expression, the first right-hand term reduces to zero and the summation terms go by steps of π/l, and for l infinite, π/l is infinitely small. If we put $n\pi/l =$ s, then $\pi/l = ds$ and the above becomes

$$i = \frac{2E}{\pi R}\int_0^\infty \epsilon^{-\frac{s^2}{RC}t}\cos(sx)ds. \qquad (28)$$

For $x = 0$ at the transmitting end, this reduces to

$$i = \frac{2E}{\pi R}\int_0^\infty \epsilon^{-\frac{s^2}{RC}t}\,ds.$$

Put

$$\frac{s^2}{RC}t = y^2;\ \frac{s\sqrt{t}}{\sqrt{RC}} = y;\ ds = \sqrt{\frac{RC}{t}}dy,$$

and

$$i = \frac{2E\sqrt{RC}}{\pi R\sqrt{t}}\int_0^\infty \epsilon^{-y^2}dy.$$

This is a well-known definite integral whose value is $\sqrt{\pi}/2$ (see Osgood's "Advanced Calculus."

Hence,

$$i = \frac{2E\sqrt{RC}}{\pi R\sqrt{t}}\frac{\sqrt{\pi}}{2} = E\sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}}. \tag{29}$$

Comparing this with the second equation of (7), which is also an expression for the current at the transmitting end of an infinite cable, we get the relation

$$\sqrt{\frac{C}{R}p} = \sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}},$$

and

$$\sqrt{p} = \frac{1}{\sqrt{\pi t}}; \tag{30}$$

which establishes, at least in an experimental way, the validity of the assumption concerning the fractional differential operator. Another proof is given in Chap. V.

Cable Open at One End.—Another simple case is that of a cable open at the far end. The terminal conditions for this case are

$$x = 0;\ V = E,$$
$$x = l;\ i = 0,$$

which give, by (16),

$$\left.\begin{aligned} A + B &= E, \\ -A\epsilon^{Kl} + B\epsilon^{-Kl} &= 0. \end{aligned}\right\} \tag{31}$$

Hence,

$$\left.\begin{aligned} A &= \frac{E\epsilon^{-Kl}}{\epsilon^{Kl} + \epsilon^{-Kl}}, \\ B &= \frac{E\epsilon^{Kl}}{\epsilon^{Kl} + \epsilon^{-Kl}}. \end{aligned}\right\} \tag{32}$$

Introducing these values in (16), we obtain the equations for the voltage and current as follows:

$$\left.\begin{aligned} V &= E\frac{\cosh K(l-x)}{\cosh Kl}, \\ i &= E\sqrt{\frac{Cp}{R}}\frac{\sinh K(l-x)}{\cosh Kl}. \end{aligned}\right\} \tag{33}$$

Applying the expansion theorem, we have, for this case, the determinantal equation

$$Z(p) = \cosh Kl = 0, \tag{34}$$

which gives

$$Kl = j\frac{n\pi}{2};\ n = 1, 3, 5, 7, \cdots$$

and

$$K = \sqrt{RCp} = j\frac{n\pi}{2l}.$$

From this we get

$$p = -\frac{n^2\pi^2}{4RCl^2}. \tag{35}$$

Also,

$$\frac{\partial Z(p)}{\partial p} = \sinh Kl\frac{\partial(Kl)}{\partial p},$$

but

$$\frac{\partial(Kl)}{\partial p} = \frac{RCl^2}{jn\pi},$$

hence,

$$\frac{\partial Z(p)}{\partial p} = \frac{RCl^2}{jn\pi}\sinh Kl = \frac{RCl^2}{n\pi}\sin\frac{n\pi}{2}. \tag{36}$$

For

$$p = 0;\ \cosh K(l-x) = 1 = \cosh Kl.$$

Introducing these values in the expansion formula, we obtain the developed solution for the voltage as follows:

$$V = E - E\sum\frac{\cos\frac{n\pi}{2}\left(1-\frac{x}{l}\right)\epsilon^{-\frac{n^2\pi^2}{4RCl^2}t}}{\frac{n^2\pi^2}{4RCl^2}\cdot\frac{RCl^2}{n\pi}\sin\frac{n\pi}{2}},$$

$$= E - \frac{4E}{\pi}\sum\frac{1}{n}\sin\frac{n\pi x}{2l}\epsilon^{-\frac{n^2\pi^2}{4RCl^2}t}. \tag{37}$$

The expression for the current can be readily derived from (37) by the relation $i = -1/R\, dV/dx$.

Thus:

$$i = \frac{2E}{Rl}\sum \cos\frac{n\pi x}{2l}\epsilon^{-\frac{n^2\pi^2}{4RCl^2}t}. \tag{38}$$

To show the application of the second formula of the expansion theorem, derived for the case of applied alternating voltage, we shall consider the preceding problem, assuming an alternating voltage of sine wave form and of frequency $f = \omega/2\pi$ applied at the transmitting end. For this case, we have the formula

$$V = \frac{E\epsilon^{j\omega t}}{Z(p)_{p=j\omega}} + E\sum\frac{\epsilon^{p_n t}}{(p_n - j\omega)\frac{\partial Z(p)}{\partial p}p = p_n}; \tag{39}$$

only the real part of the above to be taken.

The determinantal equation is the same as in the preceding problem of applied steady voltage, and we have, therefore, the same values of p and $\partial Z(p)/\partial p$. The summation term is to be modified only by replacing p_n by $p_n - j\omega$. For the first right-hand term, we have

$$\frac{1}{Z(p)_{p=j\omega}} = \frac{\cosh K(l-x)}{\cosh Kl} = \frac{\cosh\sqrt{j\omega RC}(l-x)}{\cosh\sqrt{j\omega RC}l},$$

$$= \frac{\cosh\left\{\frac{\sqrt{RC\omega}}{2}(1+j)(l-x)\right\}}{\cosh\left\{\frac{\sqrt{RC\omega}}{2}(1+j)l\right\}}$$

Hence,

$$V = E\epsilon^{j\omega t}\frac{\cosh\left\{\frac{\sqrt{Rc\omega}}{2}(1+j)(l-x)\right\}}{\cosh\left\{\frac{\sqrt{Rc\omega}}{2}(1+j)l\right\}} +$$

$$E\sum\frac{p_n\epsilon^{p_n t}\cosh K(l-x)}{(p_n^2+\omega^2)\frac{\partial z(p)}{\partial p}p = p_n}. \tag{40}$$

Introducing the values of p_n and $\partial Z(p)/\partial p$ from (35) and (36), equation (40) takes the form

$$V = E\epsilon^{j\omega t}\frac{\cosh\left\{\frac{\sqrt{RC\omega}}{2}(1+j)(l-x)\right\}}{\cosh\left\{\frac{\sqrt{RC\omega}}{2}(1+j)l\right\}} - \frac{E\pi^3}{4RCl^2}\sum\frac{n^3\sin\frac{n\pi x}{2l}\epsilon^{-\frac{n^2\pi^2}{4RCl^2}t}}{\frac{n^4\pi^4}{16RCl^2}+\omega^2}. \qquad (41)$$

For $\omega = 0$ (41) reduces to (37), as it should.

Voltage Applied at Intermediate Point on Cable.—We shall now consider a more comprehensive case, that of applied voltage E at an intermediate point on the cable. We have, in this case, two series of waves, the primary wave is doubled, going both

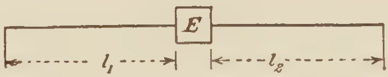

FIG. 17.

ways; each wave suffers reflection at both ends of the cable. Count distance from the end to the left of E the voltage applied at the point $x = l_1$, and the total length of the cable is l. Designate by V_1, i_1 and V_2, i_2 the voltage and current to the right and left of E, respectively.

For the right of E,

$$\left.\begin{aligned} V_1 &= A_1\epsilon^{K(x-l_1)} + B_1\epsilon^{-K(x-l_1)}, \\ i_1 &= \frac{K}{R}\{-A_1\epsilon^{K(x-l_1)} + B_1\epsilon^{-K(x-l_1)}\}. \end{aligned}\right\} \qquad (42)$$

For the left of E,

$$\left.\begin{aligned} V_2 &= A_2\epsilon^{Kx} + B_2\epsilon^{-Kx}, \\ i_2 &= \frac{K}{R}(A_2\epsilon^{Kx} - B_2\epsilon^{-Kx}). \end{aligned}\right\} \qquad (43)$$

The distances to the right and left of E are counted in opposite directions, hence, the reversal of sign in i_2.

For a cable open at both ends, we have the conditions, when

$$x = l;\ i_1 = 0,$$
$$x = 0;\ i_2 = 0.$$

Hence,

$$-A_1\epsilon^{K(l-l_1)} + B_1\epsilon^{-(Kl-l_1)} = 0,$$
$$A_2 - B_2 = 0,$$

and

$$\left.\begin{aligned} B_1 &= A_1\epsilon^{2K(l-l_1)}, \\ B_2 &= A_2. \end{aligned}\right\} \tag{44}$$

At $x = l_1$, the point of voltage application, the following conditions must hold:

$$i_1 = -i_2,$$
$$V_1 - V_2 = E,$$

which give the following:

$$-A_1 + B_1 = -A_2\epsilon^{Kl_1} + B_2\epsilon^{-Kl_1},$$
$$A_1 + B_1 - A_2\epsilon^{Kl_1} - B_2\epsilon^{-Kl_1} = E.$$

Introducing the values of B_1 and B_2 from (44), we get

$$-A_1 + A_1\epsilon^{2K(l-l_1)} + A_2\epsilon^{Kl_1} - A_2\epsilon^{-Kl_1} = 0,$$
$$A_1 + A_1\epsilon^{2K(l-l_1)} - A_2\epsilon^{Kl_1} - A_2\epsilon^{-Kl_1} = E,$$

or

$$A_1\{\epsilon^{K(l-l_1)} - \epsilon^{-K(l-l_1)}\}\epsilon^{K(l-l_1)} + A_2\{\epsilon^{Kl_1} - \epsilon^{-Kl_1}\} = 0,$$
$$A_1\{\epsilon^{K(l-l_1)} + \epsilon^{-K(l-l_1)}\}\epsilon^{K(l-l_1)} - A_2\{\epsilon^{Kl_1} + \epsilon^{-Kl_1}\} = E,$$

which, on solving, give the following values for A_1 and A_2:

$$A_1 = \frac{E\epsilon^{-K(l-l_1)}\{\epsilon^{Kl_1} - \epsilon^{-Kl_1}\}}{\{\epsilon^{K(l-l_1)} - \epsilon^{-K(l-l_1)}\}\{\epsilon^{Kl_1} + \epsilon^{-Kl_1}\} + \{\epsilon^{K(l-l_1)} + \epsilon^{-K(l-l_1)}\}\{\epsilon^{Kl_1} - \epsilon^{-Kl_1}\}}$$

$$\left.\begin{aligned} &= \frac{E\epsilon^{-K(l-l_1)}\sinh Kl_1}{2\sinh Kl}, \\ A_2 &= \frac{-E\sinh K(l-l_1)}{\sinh Kl}. \end{aligned}\right. \tag{45}$$

Substituting these values in (42) and (43), we obtain, by simple transformation, the following expressions for V_1 and V_2, the voltages to the right and left of the point of the applied voltage:

Thus:

$$\left.\begin{aligned} V_1 &= \frac{E \sinh Kl_1}{\sinh Kl} \cosh K(l - x), \\ V_2 &= \frac{-E \sinh K(l - l_1)}{\sinh Kl} \cosh Kx. \end{aligned}\right\} \tag{46}$$

In a similar way, we can obtain expressions for the voltages on the cable when both ends are grounded or when one end is grounded and the other end open—two cases.

The two ends grounded:

$$\left.\begin{aligned} V_1 &= \frac{E \cosh Kl_1}{\sinh Kl} \sinh K(l - x), \\ V_2 &= \frac{-E \cosh K(l - l_1)}{\sinh Kl} \sinh Kx. \end{aligned}\right\} \tag{47}$$

The end to the left grounded, and the end to the right open:

$$\left.\begin{aligned} V_1 &= \frac{E \cosh Kl_1}{\cosh Kl} \cosh K(l - x), \\ V_2 &= \frac{-E \sinh K(l - l_1)}{\cosh Kl} \sinh Kx. \end{aligned}\right\} \tag{48}$$

The end to the right grounded, and the end to the left open:

$$\left.\begin{aligned} V_1 &= \frac{E \sinh Kl_1}{\cosh Kl} \sinh K(l - x), \\ V_2 &= \frac{-E \cosh K(l - l_1)}{\cosh Kl} \cosh Kx. \end{aligned}\right\} \tag{49}$$

Expressions for the currents can be readily derived from the voltage expressions by the relation

$$i = -\frac{1}{R}\frac{dV}{dx}.$$

For either of the above cases, the complete solution is readily developed by the application of the expansion theorem. It will

be sufficient to consider only one case, and we shall take that of cable open at both ends for which formulas (46) apply. The determinantal equation is

$$Z(p) = \sinh Kl = 0,$$
$$Kl = jn\pi;\ n = 1, 2, 3, \cdots$$

The values of p corresponding to the roots of this equation are given by

$$K = \sqrt{RCp} = j\frac{n\pi}{l}$$

and

$$p_n = \frac{-n^2\pi^2}{RCl^2}. \tag{50}$$

$$\frac{\partial Z(p)}{\partial p} = \cosh Kl\frac{\partial(Kl)}{\partial p} = \frac{RCl^2}{2jn\pi}\cosh Kl,$$
$$= \frac{RCl^2}{2jn\pi}\cos(n\pi). \tag{51}$$

For $p = 0$,

$$\cosh K(l - x) = 1$$

$$\frac{\sinh Kl_1}{\sinh Kl} = \frac{Kl_1}{Kl} = \frac{l_1}{l}.$$

Introducing these values in the expansion-theorem formula, we get

$$V_1 = \frac{El_1}{l} + E\sum\frac{j\sin\frac{n\pi l_1}{l}\cos n\pi\left(1 - \frac{x}{l}\right)\epsilon^{-\frac{n^2\pi^2}{RCl^2}t}}{-\frac{n^2\pi^2}{RCl^2}\cdot\frac{RCl^2}{2jn\pi}\cos(n\pi)},$$

which simplifies to the following:

$$V_1 = E\frac{l_1}{l} + \frac{2E}{\pi}\sum\frac{1}{n}\sin\frac{n\pi l_1}{l}\cos\frac{n\pi x}{l}\epsilon^{-\frac{n^2\pi^2}{RCl^2}t}.$$

Put $n\pi/l = s_1$, and the above takes the form,

$$V_1 = \frac{El_1}{l} + 2E\sum\frac{\sin(sl_1)\cos(sx)\epsilon^{-\frac{s^2}{RC}t}}{sl}. \tag{52}$$

For V_2 we use the same values of p and $\partial Z(p)/\partial p$, but in this case, for $p = 0$, $\dfrac{1}{Z(p)_{p=0}} = \dfrac{K(l - l_1)}{Kl} = 1 - \dfrac{l_1}{l}.$

Hence,

$$-V_2 = E\left(1 - \frac{l_1}{l}\right) + E\sum \frac{j \sin n\pi\left(1 - \frac{l_1}{l}\right) \cos \frac{n\pi x}{l}}{\frac{-n^2\pi^2}{RCl^2} \cdot \frac{RCl^2}{2jn\pi} \cos (n\pi)} \epsilon^{-\frac{n^2\pi^2 t}{RCl^2}},$$

$$= E\left(1 - \frac{l_1}{l}\right) - 2E\sum \frac{\sin n\pi\frac{l_1}{l} \cos \frac{n\pi x}{l}}{n\pi} \epsilon^{-\frac{n^2\pi^2}{RCl^2}t},$$

and for $n\pi/l = s$,

$$V_2 = -E\left(1 - \frac{l_1}{l}\right) + 2E\sum \frac{\sin (sl_1) \cos (sx)}{sl} \epsilon^{-\frac{s^2}{RC}t}. \tag{53}$$

For cable grounded at both ends, formula (47), the determinantal equation is the same as in the previous case. Hence, for this case, also,

$$p_n = \frac{-n^2\pi^2}{RCl^2}; \frac{\partial Z(p)}{\partial p} = \frac{RCl^2}{2jn\pi} \cos (n\pi).$$

For $p = 0$, we have, for V_1,

$$\frac{1}{Z(p)_{p=0}} = \frac{l - x}{l} = 1 - \frac{x}{l},$$

and for V_2,

$$\frac{1}{Z(p)_{p=0}} = \frac{x}{l}.$$

Combining and simplifying, we obtain

$$V_1 = E\left(1 - \frac{x}{l}\right) - 2E\sum \frac{\cos \frac{n\pi l_1}{l} \sin \frac{n\pi x}{l}}{n\pi} \epsilon^{-\frac{n^2\pi^2}{RCl^2}t},$$

$$= E\left(1 - \frac{x}{l}\right) - 2E\sum \frac{\cos (sl_1) \sin (sx)}{sl} \epsilon^{-\frac{s^2}{RC}t}, \tag{54}$$

and

$$V_2 = \frac{-Ex}{l} - 2E\sum \frac{\cos (sl) \sin (sx)}{sl} \epsilon^{-\frac{s^2}{RC}t}. \tag{55}$$

The complete solutions for the other two cases, (48) and (49), can be developed in the same way.

Cable with Terminal Impedances.—We have so far confined the discussion to a free cable, that is, a cable with free ends, either grounded or open, in which case the reflected waves are copies of the incident waves, with or without reversal. The series of waves resulting from repeated reflections are all of the

same character, which makes it possible to express the solutions in a comparatively simple form. When, however, impedances are introduced at the terminals, the reflections are no longer complete, and the problem assumes a more complicated character. In some cases, however, it is possible, even for the condition of impedances at the terminals, to obtain solutions either by direct operational process or by the application of the expansion theorem.

We shall consider first the case of an infinite cable, an impedance Z_0 at the transmitting end. The terminal condition for this case is

$$x = 0;\ V_0 + Z_0 i_0 = E. \tag{56}$$

By (3) and (5),

$$B + Z_0\sqrt{\frac{Cp}{R}}B = E,$$

and

$$B = \frac{E}{1 + Z_0\sqrt{\frac{Cp}{R}}}.$$

Hence,

$$V = \frac{E}{1 + Z_0\sqrt{\frac{Cp}{R}}}\epsilon^{-Kx}. \tag{57}$$

We shall investigate the input voltage, that is, the voltage at the transmitting end, for different types of impedances.

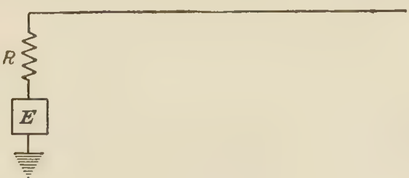

FIG. 18.

Resistance at Terminal; $Z_0 = R_0$.—For this case,

$$V_0 = \frac{E}{1 + R_0\sqrt{\frac{Cp}{R}}}.$$

By division,

$$V_0 = E\left\{1 - R_0\left(\frac{Cp}{R}\right)^{\frac{1}{2}} + R_0^2\left(\frac{Cp}{R}\right) - R^3\left(\frac{Cp}{R}\right)^{\frac{3}{2}} + \cdots\right\}. \tag{58}$$

The terms of even powers of p imply complete differentiation of unity, which would give zero, of course, and are, therefore, to be disregarded, and the above reduces to

$$V_0 = E - ER_0\left\{1 + R_0^2\left(\frac{Cp}{R}\right) + R_0^4\left(\frac{Cp}{R}\right)^2 + \cdots\right\}\left(\frac{Cp}{R}\right)^{\frac{1}{2}}. \quad (59)$$

Operating by $p^{\frac{1}{2}}$ on unity gives $1/\sqrt{\pi t}$ by (8); hence,

$$V_0 = E - ER_0\left(\frac{CR}{\pi}\right)^{\frac{1}{2}}\left\{1 + R_0^2\left(\frac{Cp}{R}\right) + R_0^4\left(\frac{Cp}{R}\right)^2 + \cdots\right\}\frac{1}{\sqrt{t}}. \quad (60)$$

Operating on $1/\sqrt{t}$ by p, p^2, etc., involves repeated differentiation, which can be done at sight (see (12)) and we get the complete solution

$$V_0 = E - ER_0\left(\frac{C}{R\pi t}\right)^{\frac{1}{2}}\left\{1 - \left(\frac{R_0^2C}{2Rt}\right) + 1.3\left(\frac{R_0^2C}{2Rt}\right)^2 - \cdots\right\}. \quad (61)$$

This solution is in the form of a series in descending powers of t, suitable for computation for large values of t. For very small values of t, the above formula is obviously not well adapted. It is possible, however, to develop (58) in the form of a series of ascending powers of t convenient for computation when t is small.

Write equation (58) in this form:

$$V_0 = \frac{E\left(\dfrac{R}{R_0^2Cp}\right)^{\frac{1}{2}}}{1 + \left(\dfrac{R}{R_0^2Cp}\right)^{\frac{1}{2}}}. \quad (62)$$

By division,

$$V_0 = E\left\{1 - \left(\frac{R}{R_0^2Cp}\right)^{\frac{1}{2}} + \left(\frac{R}{R_0^2Cp}\right) - \left(\frac{R}{R_0^2Cp}\right)^{\frac{3}{2}} + \cdots\right\}\left(\frac{R}{R_0^2Cp}\right)^{\frac{1}{2}}.$$

This may be put in the following form:

$$V_0 = E\left\{\frac{R}{R_0^2Cp} + \left(\frac{R}{R_0^2Cp}\right)^2 + \left(\frac{R}{R_0^2Cp}\right)^3 + \cdots\right\}\left(\frac{R_0^2Cp}{p}\right)^{\frac{1}{2}}$$
$$- E\left\{\frac{R}{R_0^2Cp} + \left(\frac{R}{R_0^2Cp}\right)^2 + \cdots\right\}.$$

In the first series of the above equation, the operation required is successive integrations of $p^{\frac{1}{2}} = 1/\sqrt{\pi t}$; and in the second series, the operation required is successive integration of unity.

In either case, the operations can be performed at sight, bearing in mind that

$$\frac{1}{p}t^{\frac{1}{2}} = 2t^{\frac{1}{2}},$$

$$\frac{1}{p^2}t^{\frac{1}{2}} = \frac{2\cdot 2}{1\cdot 3}t^{\frac{3}{2}},$$

$$\frac{1}{p^3}t^{\frac{1}{2}} = \frac{2\cdot 2\cdot 2}{1\cdot 3\cdot 5}t^{\frac{5}{2}}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Performing these operations, we get the following:

$$V_0 = \left\{\frac{2R}{R_0^2C}t^{\frac{1}{2}} + \left(\frac{2R}{R_0^2C}\right)^2\frac{t^{\frac{3}{2}}}{1\cdot 3} + \left(\frac{2R}{R_0^2C}\right)^3\frac{t^{\frac{5}{2}}}{1\cdot 3\cdot 5} + \cdot\cdot\cdot\right\}\left(\frac{R_0C}{R\pi}\right)^{\frac{1}{2}}$$
$$- E\left\{\frac{Rt}{R_0^2C} + \left(\frac{R}{R_0^2C}\right)^2\frac{t^2}{1\cdot 2} + \left(\frac{R}{R_0^2C}\right)^3\frac{t^3}{1\cdot 2\cdot 3}\cdot\cdot\cdot\right\}$$

and this simplifies to

$$V_0 = 2E\left(\frac{Rt}{R_0^2C\pi}\right)^{\frac{1}{2}}\left\{1 + \frac{2R}{R_0^2C}\frac{t}{3} + \frac{1}{3\cdot 5}\left(\frac{2Rt}{R_0^2C}\right)^2 + \cdot\cdot\cdot\right\} + E\left(1 - \epsilon^{\frac{Rt}{R_0^2C}}\right), \quad (63)$$

a very convenient formula for calculating V_0 for small values of t. Formulas (61) and (63) supplement each other, one to be used for large values of t and the other for small values of t.

Terminal Impedance $z_0 = 1/c_0p$.—Substituting $1/C_0p$ for Z_0 in (57), we get the expression for V_0 for this condition,

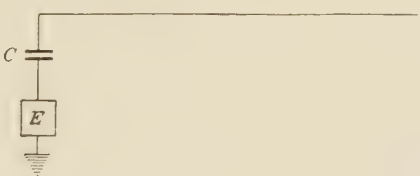

Fig. 19.

$$V_0 = \frac{E}{1 + \frac{1}{C_0p}\sqrt{\frac{Cp}{R}}} = \frac{E}{1 + \sqrt{\frac{C}{C_0^2Rp}}}.$$

Put for brevity $C/C_0^2R = 1/a$, and

$$V_0 = \frac{E}{1 + \frac{1}{\sqrt{ap}}}. \quad (64)$$

By division, the above expands in the series

$$V = E\left\{1 - \frac{1}{(ap)^{\frac{1}{2}}} + \frac{1}{(ap)} - \frac{1}{(ap)^{\frac{3}{2}}} + \cdots\right\}$$

$$= E\left\{1 + \frac{1}{ap} + \frac{1}{(ap)^2} + \frac{1}{(ap)^3} + \cdots\right\}$$

$$- E\left\{1 + \frac{1}{ap} + \frac{1}{(ap)^2} + \cdots\right\}\frac{1}{(ap)^{\frac{1}{2}}}. \tag{65}$$

The first bracket term of (65) involves complete integration of unity-time function which can be effected at sight. Thus:

$$E\left\{1 + \frac{1}{ap} + \frac{1}{(ap)^2} + \cdots\right\} = E\left\{1 + \frac{t}{a} + \frac{1}{2!}\left(\frac{t}{a}\right)^2 + \cdots\right\}$$

$$= E\epsilon^{\frac{t}{a}}. \tag{66}$$

To evaluate the second bracket term of (65) involves integrations of the factor $1/(ap)^{\frac{1}{2}}$.

Now,

$$\frac{1}{p^{\frac{1}{2}}} = \frac{1}{p}p^{\frac{1}{2}} = \frac{1}{p}\frac{1}{\sqrt{\pi t}} = \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}}$$

$$\frac{1}{p}\frac{1}{p^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}}\int_0^t t^{\frac{1}{2}}dt = \frac{2 \cdot 2}{3\sqrt{\pi}}t^{\frac{3}{2}}$$

$$\frac{1}{p^2}\frac{1}{p^{\frac{1}{2}}} = \frac{2 \cdot 2}{3\sqrt{\pi}}\int_0^t t^{\frac{3}{2}}dt = \frac{2 \cdot 2 \cdot 2}{3 \cdot 5\sqrt{\pi}}t^{\frac{5}{2}}.$$

. . . .

Hence,

$$E\left\{1 + \frac{1}{ap} + \frac{1}{(ap)^2} + \frac{1}{(ap)^3} + \cdots\right\}\frac{1}{(ap)^{\frac{1}{2}}} =$$

$$2E\left(\frac{t}{\pi a}\right)^{\frac{1}{2}}\left\{1 + \frac{1}{3}\left(\frac{2t}{a}\right) + \frac{1}{3 \cdot 5}\left(\frac{2t}{a}\right)^2 + \cdots\right\} \tag{67}$$

Combining the results of (66) and (67), we obtain the complete developed solution for the voltage at the transmitting end,

$$V_0 = E\epsilon^{\frac{t}{a}} - 2E\left(\frac{t}{\pi a}\right)^{\frac{1}{2}}\left\{1 + \frac{1}{3}\frac{2t}{a} + \frac{1}{3 \cdot 5}\left(\frac{2t}{a}\right)^2 + \cdots\right\} \tag{68}$$

a very convenient formula for computation when t is small, the series rapidly convergent. Heaviside gave an alternate formula for this case, adapted for large values of t, which is as follows:[1]

$$V = E\left(\frac{a}{\pi t}\right)^{\frac{1}{2}}\left\{1 - \frac{a}{2t} + 1 \cdot 3\left(\frac{a}{2t}\right)^2 + \cdots\right\}. \tag{69}$$

CABLE OF FINITE LENGTH; TERMINAL IMPEDANCES

When we depart from the ideal condition of a free cable, the ends either open or grounded, we run into considerable mathematical difficulties. The problem becomes more difficult because of the fact that the reflections at the ends are no longer complete; part of the wave energy is absorbed by the terminal apparatus, and only part of it is reflected, with the result that at each reflection the entire character of the wave is changed. The final solution must, of course, sum up all the repeated reflections at each end, which introduces considerable complexity in any attempt to obtain a general solution. In some special cases, however, it is possible to obtain a solution in a form suitable for numerical calculations.

It would be well worth while for the student who desires to obtain a mental picture of the physical process in the building up of wave solutions to read Heaviside's "Electromagnetic Theory," Vol. II, pp. 67 to 77. He sets forth clearly the whole process, starting out with a single wave generated at one end, taking proper account of its attenuation as it reaches the other end, multiplying by a coefficient of reflection, and transmitted back on the line; again attenuated and reflected, the process repeated over and over. Summing all up, he arrives in a physical way at the complete expression for the voltage on the cable.

Let us assume a cable of length l; impedances z_t and z_r connected at the transmitting and receiving ends, respectively. We have the general expressions for the voltage and current, equations (3) and (5), which are applicable for all cases, irrespective of the terminal conditions.

$$\left.\begin{aligned} V &= A\epsilon^{Kx} + B\epsilon^{-Kx}, \\ i &= \frac{K}{R}(-A\epsilon^{Kx} + B\epsilon^{-Kx}), \\ K &= \sqrt{RCp}. \end{aligned}\right\} \tag{70}$$

[1] See Heaviside, "Electromagnetic Theory," Vol. II, p. 42.

A and B are determined from the terminal conditions, which are as follows:

$$\left.\begin{aligned} &\text{For } x = 0;\ V_0 = E - z_t i_0, \\ &\text{For } x = l;\ V_l = z_r i_l. \end{aligned}\right\} \tag{71}$$

These give, on substitution in (69),

$$\left.\begin{aligned} A + B &= E - \frac{z_t K}{R}(-A + B), \\ A\epsilon^{Kl} + B\epsilon^{-Kl} &= \frac{z_r K}{R}(-A\epsilon^{Kl} + B\epsilon^{-Kl}). \end{aligned}\right\}$$

Rearranging,

$$\left.\begin{aligned} A\left(1 - \frac{z_t K}{R}\right) + B\left(1 + \frac{z_t K}{R}\right) &= E, \\ A\epsilon^{Kl}\left(1 + \frac{z_r K}{R}\right) + B\epsilon^{-Kl}\left(1 - \frac{z_r K}{R}\right) &= 0. \end{aligned}\right\} \tag{72}$$

From these two equations, expressions for A and B are readily obtained:

$$\left.\begin{aligned} A &= \frac{E\left(1 - \dfrac{z_r K}{R}\right)\epsilon^{-Kl}}{\left(1 + z_r z_t \dfrac{K^2}{R^2}\right)(\epsilon^{-Kl} - \epsilon^{Kl}) - \dfrac{K}{R}(z_t + z_r)(\epsilon^{Kl} + \epsilon^{-Kl})}, \\ B &= \frac{E\left(1 + \dfrac{z_r K}{R}\right)\epsilon^{Kl}}{\left(1 + z_r z_t \dfrac{K^2}{R^2}\right)(\epsilon^{-Kl} - \epsilon^{Kl}) - \dfrac{K}{R}(z_t + z_r)(\epsilon^{Kl} + \epsilon^{-Kl})}. \end{aligned}\right\} \tag{73}$$

Introducing these values of A and B into (69), we obtain the following expressions for the voltage and current in the cable:

$$\left.\begin{aligned} V &= E\frac{(\epsilon^{K(l-x)} - \epsilon^{-K(l-x)}) + z_r \dfrac{K}{R}(\epsilon^{K(l-x)} + \epsilon^{-K(l-x)})}{\left(1 + z_r z_t \dfrac{K^2}{R^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + \dfrac{K}{R}(z_t + z_r)(\epsilon^{Kl} + \epsilon^{-Kl})}, \\ i &= E\frac{K}{R}\,\frac{(\epsilon^{K(l-x)} + \epsilon^{-K(l-x)}) + z_r \dfrac{K}{R}(\epsilon^{K(l-x)} - \epsilon^{-K(l-x)})}{\left(1 + z_r z_t \dfrac{K^2}{R^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + \dfrac{K}{R}(z_t + z_r)(\epsilon^{Kl} + \epsilon^{-Kl})} \end{aligned}\right\} \tag{74}$$

Expressed in hyperbolic functions,

$$\left.\begin{aligned} V &= E\frac{\sinh K(l-x) + z_r\frac{K}{R}\cosh K(l-x)}{\left(1 + z_r z_t\frac{K^2}{R^2}\right)\sinh Kl + \frac{K}{R}(z_t + z_r)\cosh Kl}, \\ i &= E\frac{K}{R}\frac{\cosh K(l-x) + z_r\frac{K}{R}\sinh K(l-x)}{\left(1 + z_r z_t\frac{K^2}{R^2}\right)\sinh Kl + \frac{K}{R}(z_t + z_r)\cosh Kl}. \end{aligned}\right\} \quad (75)$$

For $x = l$, at the receiving end,

$$\left.\begin{aligned} V_l &= \frac{Ez_r}{\left(\frac{R}{K} + z_t z_r\frac{K}{R}\right)\sinh Kl + (z_t + z_r)\cosh Kl}, \\ i_t &= \frac{E}{\left(\frac{R}{K} + z_r z_t\frac{K}{R}\right)\sinh Kl + (z_t + z_r)\cosh Kl}. \end{aligned}\right\} \quad (76)$$

The problem of greatest interest in cable telegraphy is to determine the current at the receiving end, usually designated *arrival current*. For this, we have the expression given by the second equation (76). To develop the complete solution from this expression by the application of the expansion theorem, it is necessary, in each case, to determine the roots of the determinantal equation,

$$Z(p) = \left(\frac{R}{K} + z_r z_t\frac{K}{R}\right)\sinh Kl + (z_t + z_r)\cosh Kl = 0, \quad (77)$$

or

$$\tanh Kl = \frac{-(z_t + z_r)}{\frac{R}{K} + z_r z_t\frac{K}{R}}. \quad (78)$$

When the terminal apparatus z_t and z_r are combinations of inductances and capacities, the determination of the roots of equation (77) may be a quite difficult matter. In special cases, however, the roots of the equation are readily determined, and the complete solution can be obtained. We shall consider here a few typical cases.

Resistance at Receiving End; Transmitting End Grounded.—The determinantal equation for this case simplifies to

$$\tanh Kl = -\frac{R_0}{R}K = -\frac{R_0}{Rl}Kl. \tag{79}$$

Put

$$Kl = jx,$$

and the above transforms to

$$\tan x = -\frac{R_0}{Rl}x. \tag{80}$$

The roots of this equation can be found approximately by means of a diagram plotting two curves

$$\left.\begin{aligned} y_1 &= \tan x, \\ y_2 &= -\frac{R_0}{Rl}x, \end{aligned}\right\} \tag{81}$$

the intersecting points of these curves giving the values of x, the roots of (80). In Fig. (20), tangent curves are plotted,

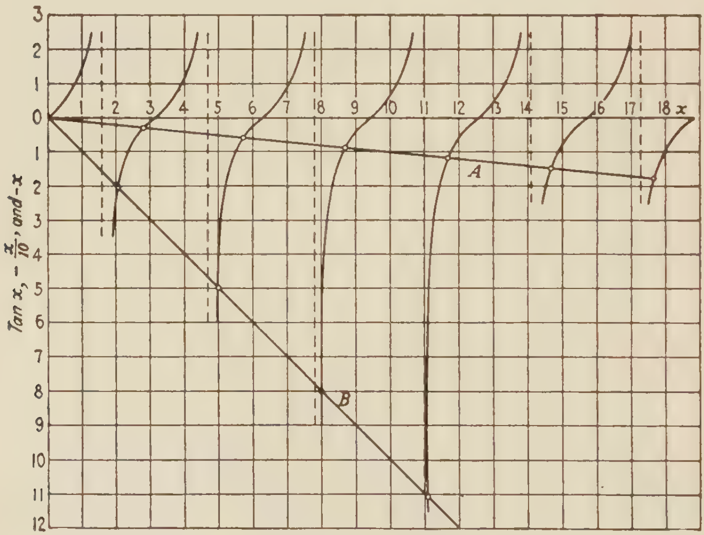

Fig. 20.

$y_1 = \tan x$, and the two lines $y_2 = -x$ and $y_2 = -x/10$, that is, for values of $R_0 = Rl$ and $R_0 = \frac{1}{10}Rl$.

The intersecting points for $y_2 = -x/10$ are

$$x = 2.8;\ 5.7;\ 8.7;\ 11.6;\ 14.8;\ 17.8.$$

For

$$y_2 = -x,$$

$$x = 2;\ 5;\ 8;\ 12.$$

These values of x are the first few roots. By extending the graphs, additional roots of the equation can be obtained.

To obtain the values of $\partial Z(p)/\partial p$, we note that, for the condition of this problem,

$$Z(p) = \frac{R}{K} \sinh Kl + R_0 \cosh Kl,$$

and

$$\frac{\partial Z(p)}{\partial p} = \left(Rl \cosh Kl - \frac{R}{K^2} \sinh Kl + R_0 l \sinh Kl\right) \frac{\partial K}{\partial p},$$

$$\frac{\partial K}{\partial p} = \partial \frac{(RCp)^{1/2}}{\partial p} = \frac{1}{2} \frac{K}{p}.$$

Hence,

$$\frac{\partial Z(p)}{\partial p} = \frac{1}{2p}\left\{Rl \cosh Kl - \frac{R}{K} \sinh Kl + R_0 Kl \sinh Kl\right\}.$$

Substituting $Kl = jx$, we have

$$p \frac{\partial Z(p)}{\partial p} = \frac{1}{2}\left(Rl \cos x - \frac{Rl}{x} \sin x - R_0 x \sin x\right). \tag{82}$$

By (80), however,

$$R_0 = -\frac{Rl}{x} \tan x,$$

and substituting this in (82), we get

$$\begin{aligned} p \frac{\partial Z(p)}{\partial p} &= \frac{1}{2}\left(Rl \cos x - \frac{Rl \sin x}{x} + Rl \frac{\sin^2 x}{\cos x}\right) \\ &= \frac{Rl}{2}\left(\frac{1}{\cos x} - \frac{\sin x}{x}\right). \end{aligned} \tag{83}$$

For $p = 0$,

$$Z(p) = \frac{RKl}{K} + R_0 = Rl + R_0. \tag{84}$$

Substituting the values from (83) and (84) in the expansion formula, we obtain the complete solution

$$i_r = \frac{E}{Rl + R_0} + 2E\sum \frac{\epsilon^{-\frac{x^2}{RCl^2}t}}{Rl\left(\frac{1}{\cos x} - \frac{\sin x}{x}\right)}, \tag{85}$$

the summation to extend for all the values of x, the roots of equation (80). When $R_0 = 0$, equation (80) gives $\tan x = 0$ or $x = n\pi$, and (85) reduces to

$$i_r = \frac{E}{Rl} + \frac{2E}{Rl}\sum \cos(n\pi)\epsilon^{-\frac{n^2\pi^2}{RCl^2}t}. \tag{86}$$

which is the same as (27) for $x = l$.

Condenser at Receiving End; $z_r = 1/c_0p$. Transmitting End Grounded; $z_t = 0$.—By (76), the determinantal equation for this case is

$$Z(p) = \frac{R}{K}\sinh Kl + \frac{1}{C_0p}\cosh Kl = 0,$$

or

$$\tanh Kl = -\frac{K}{RC_0p} = -\frac{Cl}{c_0Kl}. \tag{87}$$

Put $Kl = jx$ as in the previous case, and we have

$$\tan x = \frac{Cl}{C_0}\frac{1}{x}. \tag{88}$$

The roots of this equation are obtained in the same way as in the previous case; plotting the two curves

$$y_1 = \tan x,$$

$$y_2 = \frac{Cl}{C_0}\frac{1}{x},$$

the intersecting points of these curves giving the values of x required. The value of $\partial Z(p)/\partial p$ is readily obtained, thus:

$$\begin{aligned} p\frac{\partial Z(p)}{\partial p} &= \frac{1}{2}Rl\cosh Kl - \frac{1}{2}\frac{Rl}{Kl}\sinh Kl + \frac{1}{2}\frac{Kl}{C_0p}\sinh Kl - \frac{1}{C_0p}\cosh Kl \\ &= \frac{1}{2}Rl\cos x - \frac{1}{2}\frac{Rl}{x}\sin x + \frac{1}{2}\frac{RCl^2}{C_0x}\sin x + \frac{RCl^2}{C_0x^2}\cos x. \end{aligned} \tag{89}$$

Substituting in (89) the values of $1/C_0x = \tan x/Cl$ by (87), and it simplifies to the following:

$$p\frac{\partial Z(p)}{\partial p} = \frac{1}{2}Rl\left(\frac{1}{\cos x} + \frac{\sin x}{x}\right). \tag{90}$$

For $p = 0$, $Z(p) = \infty$ no steady-state component, since there is a condenser in series with the cable. Substituting in the expansion formula, we obtain the solution for the current at the receiving end,

$$i_r = \frac{2E}{Rl}\sum \frac{\epsilon^{-\frac{x^2}{RCl^2}t}}{\frac{1}{\cos x} + \frac{\sin x}{x}}, \tag{91}$$

the summation to extend for all values of x, the roots of the equation (88).

Condenser at Transmitting End; $z_t = 1/c_tp$. Receiving End Grounded; $z_r = 0$.—The determinantal equation by (77) is

$$Z(p) = \frac{R}{K}\sinh Kl + \frac{1}{C_tp}\cosh Kl = 0$$

or

$$\tanh Kl = -\frac{K}{RC_tp} = -\frac{Cl}{C_tKl}. \tag{92}$$

This is exactly the same form as (87), and, hence, the solution for this problem is the same as in the preceding case, except that $1/c_0$ is replaced by $1/c_t$.

Condensers at Transmitting and Receiving Ends $z_t = 1/c_tp$; $z_r = 1/c_rp$.—Substituting these values of z_t and z_r in the determinantal equation (77), we have

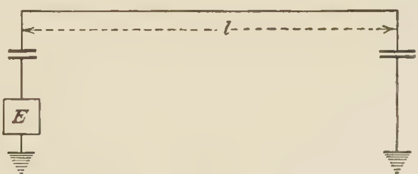

FIG. 21.

$$Z(p) = \left(\frac{R}{K} + \frac{1}{C_rC_tp^2}\frac{K}{R}\right)\sinh Kl + \left(\frac{1}{C_rp} + \frac{1}{C_tp}\right)\cosh Kl = 0, \tag{93}$$

or

$$\tanh Kl = -\frac{\dfrac{1}{C_r p} + \dfrac{1}{C_t p}}{\dfrac{R}{K} + \dfrac{1}{C_r C_t p^2}\dfrac{K}{R}}. \tag{94}$$

Introducing the value of $p = K^2/RC$, the above transforms to the following:

$$\tanh Kl = -\frac{\dfrac{RC}{K^2}(C_r + C_t)}{\dfrac{R}{K}C_r C_t + \dfrac{R^2C^2}{RK^3}}.$$

$$= -\frac{RCl^2(C_r + C_t)}{RKl^2C_rC_t + \dfrac{R^2C^2l^4}{RKl^2}}. \tag{95}$$

Put $K = jx$, and we get

$$j \tan x = \frac{RCl^2(C_r + C_t)}{Rl\left(jxC_rC_t + \dfrac{C^2l^2}{jx}\right)}$$

$$\tan x = \frac{Cl(C_r + C_t)}{xC_rC_t - \dfrac{1}{x}C^2l^2} = \frac{\dfrac{C_r + C_t}{Cl}}{x\dfrac{C_rC_t}{C^2l^2} - \dfrac{1}{x}}. \tag{96}$$

The roots of the equation can be obtained by graphical method if the ratios C_r/Cl and C_t/Cl are known; plotting the curves

$$\left.\begin{aligned} y_1 &= \tan x, \\ y_2 &= \frac{\dfrac{C_r + C_t}{Cl}}{x\dfrac{C_rC_t}{C^2l^2} - \dfrac{1}{x}}, \end{aligned}\right\} \tag{97}$$

the intersecting points giving the values of x which are the roots of equation (95).

Differentiating (93) with respect to p, we get

$$\frac{\partial Z(p)}{\partial p} = \left(-\frac{R}{K^2}\frac{\partial K}{\partial p} + \frac{1}{C_rC_t}\frac{1}{p^2}\frac{1}{R}\frac{\partial K}{\partial p} - \frac{2}{C_rC_tp^3}\frac{K}{R}\right)\sinh Kl$$
$$+ \left(\frac{R}{K} + \frac{1}{C_rC_t}\frac{1}{p^2}\frac{K}{R}\right)\cosh Kl\frac{\partial(Kl)}{\partial p} + \frac{1}{p}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\sinh Kl\frac{\partial(Kl)}{\partial p}$$
$$- \frac{1}{p^2}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\cosh Kl.$$

Introducing the values $p = K^2/RC$ and $\partial K/\partial p = RC/2K$, the above reduces to the following:

$$p\frac{\partial Z(p)}{\partial p} = Rl\left\{-\frac{1}{2Kl} - \frac{3}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{K^3l^3} + \frac{Cl}{2Kl}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right\} \sinh Kl$$
$$+ Rl\left\{\frac{1}{2} + \frac{1}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{K^2l^2} - \frac{Cl}{K^2l^2}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right\} \cosh Kl$$

Substituting $Kl = jx$, we get

$$p\frac{\partial Z(p)}{\partial p} = Rl\left\{\left[-\frac{1}{2x} + \frac{3}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{x^3} + \frac{Cl}{2x}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right] \sin x \right.$$
$$\left. + \left[\frac{1}{2} - \frac{1}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{x^2} + \frac{Cl}{x^2}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right] \cos x\right\}. \quad (98)$$

Introducing these values in the expansion formula, we obtain the complete solution for the current at the receiving end.

$$i_r = \frac{E}{Rl}\sum \frac{\epsilon^{-\frac{x^2}{RCl^2}t}}{\left\{-\frac{1}{2x} + \frac{3}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{x^3} + Cl\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right\} \sin x + \left\{\frac{1}{2} - \frac{1}{2}\frac{C^2l^2}{C_rC_t}\frac{1}{x^2} + \frac{Cl}{x^2}\left(\frac{1}{C_r} + \frac{1}{C_t}\right)\right\} \cos x} \quad (99)$$

the summation to extend for the values of x, the roots of equation (96).

When the transmitting end is grounded, $1/C_t = 0$, equation (99) reduces to that of (91), which is readily seen on comparing (98) with that of (89).

CHAPTER V

TRANSMISSION LINES

Circuits of Distributed Inductances, Capacity, Resistance, and Leakage.—In the preceding chapter, the discussion was confined to problems relating to ideal cables, negligible inductance, and leakage. We shall take up here the more general problem in which all the electrical constants of cable or line, that is, R, L, C, g are taken into account.

The relations between voltage and current at any point on the line are given by the following equations:

$$\left.\begin{aligned}(Lp + R)i &= -\frac{dV}{dx},\\ (Cp + g)V &= -\frac{di}{dx}.\end{aligned}\right\} \tag{1}$$

From these we derive the equation of propagation

$$(Lp + R)(Cp + g)V = \frac{d^2V}{dx^2}. \tag{2}$$

Put for brevity,

$$K^2 = (Lp + R)(g + Cp), \tag{3}$$

and use the following notation:

$$\left.\begin{aligned}LC &= \frac{1}{v^2},\\ \frac{R}{2L} &= a;\ \frac{g}{2C} = b,\\ \rho &= a + b;\ \sigma = a - b.\end{aligned}\right\} \tag{4}$$

We have, then,

$$K^2 = \frac{1}{v^2}(p + 2a)(p + 2b). \tag{5}$$

The solution of equation (1) is

$$V = A\epsilon^{Kx} + B\epsilon^{-Kx}. \tag{6}$$

The expression for the current is readily obtained from (6) by the aid of either equation (1), thus:

$$i = \frac{K}{Lp + R}(-A\epsilon^{Kx} + B\epsilon^{-Kx}), \tag{7}$$

and with the notation given by (4),

$$i = \frac{1}{Lv}\sqrt{\frac{p+2b}{p+2a}}(-A\epsilon^{Kx} + B\epsilon^{-Kx}). \tag{8}$$

A and B are to be determined from the terminal conditions in any given problem.

INFINITE LINE

We shall consider first the case of an infinite line, and a constant voltage E is applied at $x = 0$. A and B are readily determined as follows:

$$A = 0 \text{ and } B = E.$$

Hence,

$$i = \frac{E}{Lv}\sqrt{\frac{p+2b}{p+2a}}\epsilon^{-Kx}. \tag{9}$$

This is a symbolic solution involving the differential operator p from which the real solution is to be developed. For $x = 0$, the current at the transmitting end is given by

$$i_0 = \frac{E}{Lv}\sqrt{\frac{p+2b}{p+2a}}1. \tag{10}$$

The operand, in this case, is unity-time function. Henceforth, the unity factor will be omitted from the equations, and if no other operand, unity-time function will be implied.

To start with, we shall simplify the problem by neglecting the leakage, $g = 0$, $b = 0$, and equation (10) reduces to

$$i_0 = \frac{E}{Lv}\sqrt{\frac{p}{p+2a}}.$$

Now,

$$\left(\frac{p}{p+2a}\right)^{1/2} = \epsilon^{-at}\epsilon^{at}\left(\frac{p}{p+2a}\right)^{1/2}, \tag{11}$$

and ϵ^{at} may be shifted to the right, provided we change simultaneously p to $p - a$ (see p. 9), which makes

$$\left(\frac{p}{p+2a}\right)^{1/2} = \epsilon^{-at}\left(\frac{p-a}{p+a}\right)^{1/2}\epsilon^{at},$$

the operand changed from unity to ϵ^{at}. We have, however, shown previously that $\epsilon^{at} = \dfrac{p}{p-a}$, and making this substitution, we get

$$\left(\frac{p}{p+2a}\right)^{\frac{1}{2}} = \epsilon^{-at}\left(\frac{p-a}{p+a}\right)^{\frac{1}{2}}\frac{p}{p-a} = \epsilon^{-at}\frac{p}{(p^2-a^2)^{\frac{1}{2}}}$$
$$= \epsilon^{-at}\frac{1}{\left(1-\dfrac{a^2}{p^2}\right)^{\frac{1}{2}}}. \qquad (12)$$

The operand is again unity. Expanding (12) by the binomial theorem gives

$$\epsilon^{-at}\frac{1}{\left(1-\dfrac{a^2}{p^2}\right)^{\frac{1}{2}}} = \epsilon^{-at}\left\{1+\frac{1}{2}\frac{a^2}{p^2}+\frac{1\cdot 3}{2^2\cdot 2!}\frac{a^4}{p^4}+\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}\frac{a^6}{p^6}+\cdots\right\} \qquad (13)$$

operating on unity by each of the bracket terms involves successive integrations, that is, $1/p^n = t^n/n!$.
Hence,

$$\epsilon^{-at}\frac{1}{\left(1-\dfrac{a^2}{p^2}\right)^{\frac{1}{2}}} = \epsilon^{-at}\left\{1+\frac{a^2t^2}{2^2}+\frac{a^4t^4}{2^2\cdot 4^2}+\frac{a^6t^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right\}. \qquad (14)$$

The bracket term in the above expression is the expansion of the Bessel function of zero order, $I_0(at)$, and, therefore,

$$\epsilon^{-at}\frac{1}{\left(1-\dfrac{a^2}{p^2}\right)^{\frac{1}{2}}} = \epsilon^{-at}I_0(at). \qquad (15)$$

Substituting the values from (15) in (11), we finally obtain the expression for the current at the transmitting end of the line,

$$i_0 = \frac{E}{Lv}\epsilon^{-at}I_0(at). \qquad (16)$$

A short table of the values of $I_0(at)$ for arguments from 0 to 6 is given on page 164.

For large values of the argument, the function approaches the value given by

$$I_0(at) = \frac{\epsilon^{at}}{\sqrt{2\pi at}}.$$

For negligible inductance, a is infinitely large, the above relation holds true, and, on substitution in (16), we get

$$i_0 = \frac{E}{Lv}\frac{1}{\sqrt{2\pi at}} = \frac{E\sqrt{LC}}{L\sqrt{2\pi\frac{R}{2L}t}},$$

$$= E\sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}}. \tag{17}$$

On comparing this with equation (7), Chap. IV, the expression for the transmission current in a non-inductive cable, we obtain the relation

$$\sqrt{p}1 = \frac{1}{\sqrt{\pi t}},$$

which yields an additional proof of the validity of the operation of the fractional derivative, which was made use of extensively in the preceding chapter.

In passing, it may be observed that we have incidentally established the equivalence of the following operations, all leading on development to a series which is the zero Bessel function, namely:

$$\epsilon^{at}\left(\frac{p}{p+2a}\right)^{\frac{1}{2}} = \left(\frac{p-a}{p+a}\right)^{\frac{1}{2}}\epsilon^{at} = \frac{p}{(p^2-a^2)^{\frac{1}{2}}} =$$

$$\frac{1}{\left(1-\frac{a^2}{p^2}\right)^{\frac{1}{2}}} = I_0(at). \tag{18}$$

Leakage Included, Applied Voltage Varying as $\epsilon^{-\rho t}$.—In the more general case including leakage, we have, by (10).

$$i_0 = \frac{E}{Lv}\sqrt{\frac{p+2b}{p+2a}},$$

$$= \frac{E}{Lv}\sqrt{\frac{p+\rho-\sigma}{p+\rho+\sigma}}. \tag{19}$$

This may be transformed by multiplying by $\epsilon^{-\rho t}\epsilon^{\rho t}$, shifting $\epsilon^{\rho t}$ to the right, and changing p to $p - \rho$, which would give

$$i_0 = \frac{E}{Lv}\epsilon^{-\rho t}\sqrt{\frac{p-\sigma}{p+\sigma}}\epsilon^{\rho t}. \tag{20}$$

If we should take the applied voltage to vary as $\epsilon^{-\rho t}$, the operand would be changed to unity and the above reduce to

$$i_0 = \frac{E}{Lv}\epsilon^{-\rho t}\left(\frac{p-\sigma}{p+\sigma}\right)^{\frac{1}{2}}$$

$$= \frac{E}{Lv}\epsilon^{-\rho t}\left(1 - \frac{\sigma}{p}\right)\left(1 - \frac{\sigma^2}{p^2}\right)^{-\frac{1}{2}}.$$

By (18),

$$\left(1 - \frac{\sigma^2}{p^2}\right)^{-\frac{1}{2}} = I_0(\sigma t),$$

hence,

$$i_0 = \frac{E}{Lv}\epsilon^{-\rho t}\left(1 - \frac{\sigma}{p}\right)I_0(\sigma t). \tag{21}$$

To obtain the complete solution requires operating by $1/p$ on $I_0(\sigma t)$, that is, integrating from 0 to t. If we use the expanded form of $I_0(\sigma t)$ given by (14), the terms of which are readily integrated, we obtain the following:

$$i_0 = \frac{E}{Lv}\epsilon^{-\rho t}\left\{ I_0(\sigma t) \cdot - \left[\sigma t + \frac{\sigma^3 t^3}{2 \cdot 3!} + \frac{1 \cdot 3\sigma^5 t^5}{2 \cdot 4 \cdot 5!} + \cdot \cdot \cdot \right]\right\}. \tag{22}$$

For $t = 0$, the initial current is E/Lv, and the final current is zero, obviously, because the voltage falls to zero.

All Electrical Constants, L, R, C, g Active; Steady Voltage.—To derive the expression for the initial current in a line, all electrical constants active, under the application of a steady voltage, we proceed in the following manner:

$$i_0 = \frac{E}{Lv}\left(\frac{p+2b}{p+2a}\right)^{\frac{1}{2}},$$

$$= \frac{E}{Lv}\frac{(p+2b)}{(p+2b)^{\frac{1}{2}}(p+2a)^{\frac{1}{2}}},$$

$$= \frac{E}{Lv}\frac{p\left(1 + \frac{g}{Cp}\right)}{\{(p+\rho)^2 - \sigma^2\}^{\frac{1}{2}}}. \tag{23}$$

The development of $p/\{(p+\rho)^2 - \sigma^2\}^{1/2}$ is readily obtained by transforming it to one of the known forms given by (18), thus:

$$\frac{p}{\{(p+\rho)^2-\sigma^2\}^{1/2}} = \epsilon^{\rho t}\epsilon^{-\rho t}\frac{p}{\{(p+\rho)^2-\sigma^2\}^{1/2}} = \frac{\epsilon^{-\rho t}(p-\rho)\epsilon^{\rho t}}{\{p^2-\sigma^2\}^{1/2}},$$

$$= \epsilon^{-\rho t}\frac{p-\rho}{(p^2-\sigma^2)^{1/2}}\frac{p}{p-\rho} = \frac{\epsilon^{-\rho t}p}{(p^2-\sigma^2)^{1/2}} =$$

$$\frac{\epsilon^{-\rho t}}{\left(1-\dfrac{\sigma^2}{p^2}\right)^{1/2}} = \epsilon^{-\rho t}I_0(\sigma t). \quad (24)$$

Introducing this in (23), we obtain

$$i_0 = \frac{E}{Lv}\left(1+\frac{g}{Cp}\right)\epsilon^{-\rho t}I_0(\sigma t). \quad (25)$$

The initial current due to a steady voltage is thus expressed in terms of a known function and its time integral, meaning, of course, integration from 0 to t. Heaviside indicated several methods for effecting this integration, all leading, of course, to the same result. Shifting $\epsilon^{-\rho t}$ to the left by changing p to $p-\rho$, we may write:

$$\frac{1}{p}\epsilon^{-\rho t}I_0(\sigma t) = \frac{\epsilon^{-\rho t}}{\rho}\frac{\rho}{p-\rho}I_0(\sigma t) = \frac{\epsilon^{-\rho t}}{\rho}\frac{1}{\dfrac{p}{\rho}-1}I_0(\sigma t). \quad (26)$$

Expand $\dfrac{1}{\dfrac{p}{\rho}-1}$, and use the expanded form of $I_0(\sigma t)$; we get

$$\frac{1}{\dfrac{p}{\rho}-1}I_0(\sigma t) = \left(\frac{\rho}{p}+\frac{\rho^2}{p^2}+\frac{\rho^3}{p^3}+\cdots\right)$$
$$\left(1+\frac{\sigma^2t^2}{2^2}+\frac{\sigma^4t^4}{2^2\cdot 4^2}+\frac{\sigma^6t^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right).$$

This is integrable by sight, giving the following result:

$$\frac{1}{\dfrac{p}{\rho}-1}I_0(\sigma t) = \rho t\left(1+\frac{\sigma^2t^2}{2^2}+\frac{\sigma^4t^4}{2^2\cdot 4^2}+\cdots\right)$$
$$+\rho^2t^2\left(\frac{1}{2}+\frac{\sigma^2t^2}{2^2\cdot 3\cdot 4}+\frac{\sigma^4t^4}{2^24^2\cdot 5\cdot 6}+\cdots\right)$$
$$+\rho^3t^3\left(\frac{1}{2\cdot 3}+\frac{\sigma^2t^2}{2^2\cdot 3\cdot 4\cdot 5}+\frac{\sigma^4t^4}{2^2\cdot 4^2\cdot 5\cdot 6\cdot 7}+\cdots\right)$$
$$+\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$= u(t), \text{ say.} \quad (27)$$

The law of formation is obvious, every bracket term being the integral of the preceding bracket term. Hence,

$$\frac{1}{p}\epsilon^{-\rho t}I_0(\sigma t) = \frac{\epsilon^{-\rho t}}{\rho}u(t). \tag{28}$$

Combining this with (25), we obtain the completely developed expression for the initial current

$$i_0 = \frac{E}{Lv}\left\{ \epsilon^{-\rho t}I_0(\sigma t) + \epsilon^{-\rho t}\frac{g}{C\rho}u(t) \right\},$$

which may be put in this form:

$$i_0 = \frac{E\epsilon^{-\rho t}}{Lv}\left\{ I_0(\sigma t) + \left(1 - \frac{\sigma}{\rho}\right)u(t) \right\}. \tag{29}$$

For $g = 0$, $\sigma = \rho = a$, and the above reduces to (16).

The Current at Any Point Distant x from the Transmitting End.—To obtain the expression for the current for any point on the line, it is necessary to operate on either of the expressions obtained above for the transmitting end current by the operator ϵ^{-Kx}.

NON-DISSIPATING LINE

We shall first take the case of a non-dissipating line free from energy losses; $R = 0$ and $g = 0$. The transmitting end current is given by

$$i_0 = \frac{E}{Lv}.$$

If E is constant, the current jumps to its full value E/Lv and travels on with the velocity v reaching any point distance x at time $t = x/v$. The current jumps to its steady value at any point distance x at time $t = x/v$. For $E = f(t)$; $i_0 = f(t)$, and

$$i_x = E\epsilon^{-Kx}f(t). \tag{30}$$

Expanding ϵ^{-Kx}, remembering that for $R = g = 0$; $K = p\sqrt{LC} = p/v$.

$$i_x = \left(1 - p\frac{x}{v} + \frac{p^2}{2!}\frac{x^2}{v^2} - \frac{p^3}{3!}\frac{x^3}{v^3} + \cdots\right)f(t). \tag{31}$$

If we compare this with Taylor's series

$$f(t + h) = \left(1 + hp + \frac{h^2p^2}{2!} + \frac{h^3p^3}{3!} + \cdots\right)f(t),$$

we see that they are equivalent, and we may, therefore, write

$$i_x = f\left(t - \frac{x}{v}\right). \tag{32}$$

Whatever the character of the current at the origin, it travels at a speed v without any change whatever. This is also true of the voltage on the line; if the voltage at the transmitting end is $Ef(t)$, the voltage at x is $Ef(t - x/v)$.

DISTORTIONLESS LINE

When the electrical constants of the line are all active but related to satisfy the condition

$$\frac{R}{L} = \frac{g}{C},$$

the line is said to be *distortionless;* that is, any current or voltage wave impressed on the line is propagated at the speed v without any deformation; the waves suffer uniform attenuation in their transit on the line but no change in form. For this condition, $a = b$, and by (9) and (5),

$$i_x = \frac{E}{Lv}\epsilon^{-Kx},$$

and

$$K = \frac{1}{v}(p + 2a).$$

If

$$E_0 = Ef(t),$$

$$i_x = \frac{E}{Lv}\epsilon^{-\frac{2a}{v}x}\epsilon^{-\frac{p}{v}x} f(t).$$

and the operation to be performed is the same as (30); hence,

$$i_x = \frac{E}{Lv}\epsilon^{-\frac{2a}{v}x} f\left(t - \frac{x}{v}\right).$$

Similarly,

$$E_x = E\epsilon^{-\frac{2a}{v}x} f\left(t - \frac{x}{v}\right).$$

Since the time of transit for the waves to reach any point x is $t = x/v$, we may write the above equations in this form:

$$\left.\begin{aligned} i_x &= \frac{E}{Lv}\epsilon^{-2atf}\left(t - \frac{x}{v}\right), \\ E_x &= E\epsilon^{-2atf}\left(t - \frac{x}{v}\right). \end{aligned}\right\} \tag{33}$$

The current and voltage are in the same phase and in constant ratio for all points on the line.

This may, perhaps, seem a little clearer by considering the steady-state condition for applied alternating voltages. If the frequency of the impressed voltage is $\omega/2\pi$, the steady-state solution for the voltage and current on the line is readily obtained by replacing p by $j\omega$ in equations (5), (6), and (9). We get, then,

$$K^2 = \frac{1}{v^2}(j\omega + 2a)(j\omega + 2b),$$

$$V = E\epsilon^{-Kx}\epsilon^{j\omega t},$$

$$i = \frac{E}{Lv}\sqrt{\frac{j\omega + 2b}{j\omega + 2a}}\epsilon^{-Kx}\epsilon^{j\omega t}.$$

For the distortionless condition, $a = b$, the above reduce to

$$K = \frac{1}{v}(j\omega + 2a),$$

$$V = E\epsilon^{-\frac{2a}{v}x}\epsilon^{j\omega\left(t - \frac{x}{v}\right)},$$

$$i = \frac{E}{Lv}\epsilon^{-\frac{2ax}{v}}\epsilon^{j\omega\left(t-\frac{x}{v}\right)}.$$

The real parts of the above equations give

$$\left.\begin{aligned} V &= E\epsilon^{-\frac{2a}{v}x}\cos\omega\left(t - \frac{x}{v}\right), \\ i &= \frac{E}{Lv}\epsilon^{-\frac{2a}{v}x}\cos\omega\left(t - \frac{x}{v}\right). \end{aligned}\right\} \tag{34}$$

The waves travel along the line at the speed v and are attenuated at the rate $\epsilon^{-2a/v}$. Since $t - \dfrac{x}{v}$ is the same for all points on the line, there is no change in phase in the waves as they travel on the line. The waves retain the same form though they are diminished in size as they travel along the line. The attenuation of the waves, in this case, is independent of frequency. Hence,

any complex wave will travel on the line without any deformation; the waves arriving at the receiving end may be reduced in size, but otherwise a fair copy of that sent out at the sending end.

Inductance, Capacity, and Resistance Active.—The current at the transmitting end is, by (16),

$$i_0 = \frac{E}{Lv}\epsilon^{-at}I_0(at), \tag{16}$$

and

$$i_x = \frac{E}{Lv}\epsilon^{-Kx}\epsilon^{-at}I_0(at). \tag{35}$$

For this case,

$$K = \frac{1}{v}\sqrt{p^2 + 2ap} = \frac{1}{v}\sqrt{(p + a)^2 - a^2}.$$

By shifting ϵ^{-at} to the left, and compensating for it by changing p to $p - a$, a process explained before, equation (35) is transformed to this:

$$i_x = \frac{E}{Lv}\epsilon^{-at}\epsilon^{-\frac{x}{v}\sqrt{p^2-a^2}}I_0(at). \tag{36}$$

Expanding the exponential factor, and rearranging, the equation takes this form

$$i_x = \frac{E}{Lv}\epsilon^{-at}\left\{\left[1 + \frac{x^2}{v^2}\frac{(p^2 - a^2)}{2!} + \frac{x^4}{v^4}\frac{(p^2 - a^2)}{4!} + \cdots\right]\right. \tag{37}$$
$$\left. - \left[1 + \frac{x^2}{v^2}\frac{(p^2 - a^2)}{3!} + \frac{x^4}{v^4}\frac{(p^2 - a^2)}{5!} + \cdots\right]\frac{x}{v}(p^2 - a^2)^{1/2}\right\} I_0(at).$$

We have, however, by (18),

$$\frac{p}{(p^2 - a^2)^{1/2}}1 = I_0(at),$$

and

$$(p^2 - a^2)^{1/2}I_0(at) = p1 = 0,$$

that is, operating by the second bracket term on $I_0(at)$ produces zero result and can be thereafter neglected. It remains therefore only to evaluate the first bracket term in (37) in its operation on $I_0(at)$, that is,

$$i_x = \frac{E}{Lv}\epsilon^{-at}\left\{1 + \frac{x^2}{v^2}\frac{(p^2 - a^2)}{2!} + \frac{x^4}{v^4}\frac{(p^2 - a^2)^2}{4!} + \cdots\right\}I_0(at). \tag{38}$$

The Bessel function satisfies the differential equation

$$\left.\begin{aligned}&\frac{d^2}{dt^2}I_0(at) + \frac{1}{t}\frac{d}{dt}I_0(at) - a^2I_0(at) = 0,\\ \text{and}\qquad&\\ &(p^2 - a^2)I_0(at) = -\frac{p}{t}I_0(at) = -\frac{a^2I_1(at)}{at}.\end{aligned}\right\} \tag{39}$$

More generally, the function satisfies the relation

$$(p^2 - a^2)\frac{I_m(at)}{(at)^m} = -\frac{(2m+1)}{t}p\frac{I_m(at)}{(at)^m}$$
$$= -(2m+1)a^2\frac{I_{m+1}(at)}{(at)^{m+1}} \tag{40[1]}$$

Hence, the following:

$$\left.\begin{aligned}&(p^2 - a^2)I_0(at) = -\frac{p}{t}I_0(at) = -a^2\frac{I_1(at)}{at}\\ &(p^2 - a^2)^2I_0(at) = -(p^2 - a^2)a^2\frac{I_1(at)}{at} = 3a^4\frac{I_2(at)}{(at)^2}\\ &(p^2 - a^2)^3I_0(at) = 3(p^2 - a^2)a^4\frac{I_2(at)}{(at)^2} = -3\cdot 5a^6\frac{I_3(at)}{(at)^3}\\ &(p^2 - a^2)^4I_0(at) = -3\cdot 5(p^2 - a^2)a^6\frac{I_3(at)}{(at)^3} = 3\cdot 5\cdot 7a^8\frac{I_4(at)}{(at)^4}\\ &\qquad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\} \tag{41}$$

Introducing these values in (38), we obtain

$$i_x = \frac{E}{Lv}\epsilon^{-at}\left\{I_0(at) - \frac{a^2x^2}{v^2}\frac{1}{2}\frac{I_1(at)}{at} + \frac{a^4x^4}{b^4}\frac{1}{2\cdot 4}\frac{I_2(at)}{(at)^2} - \right.$$
$$\left.\frac{a^6x^6}{v^6}\frac{1}{2\cdot 4\cdot 6}\frac{I_3(at)}{(at)^3} + \cdots\right\} \tag{42}$$

The bracket term of above equation is the equivalent of $I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\}$. This identity can readily be established by putting both expressions in series form, arranging and comparing terms in the series of powers of x^2. We thus arrive at the following expression for the current at any point on the line:

$$i_x = \frac{E}{Lv}\epsilon^{-at}I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\}. \tag{43}$$

[1] See note on Bessel functions, in appendix.

Comparing with (16), we see that vt is turned into $(v^2t^2 - x^2)^{1/2}$ in the I_0 function.

At the time t, the region occupied by the current extends only to the distance vt from origin. Beyond this distance, the current is zero. Equation (41), therefore, expresses a wave whose front travels at speed v. At the wave front, the current is $\frac{E}{Lv}\epsilon^{-at}$. Since the function has been tabulated, the shape of the current curve along x, at successive moments of time, can be readily calculated.

For a non-inductive cable, $L = 0$, the argument of the function in (43) approaches infinity as L approaches zero value, and the value of the I_0 function is given by

$$I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\} = \frac{\epsilon^{\frac{a}{v}(v^2t^2 - x^2)^{1/2}}}{\sqrt{2\pi\frac{a}{v}(v^2t^2 - x^2)^{1/2}}}$$

$$= \frac{\epsilon^{at\left(1 - \frac{x^2}{v^2t^2}\right)^{1/2}}}{\sqrt{2\pi at\left(1 - \frac{x^2}{v^2t^2}\right)^{1/2}}}$$

$$= \frac{\epsilon^{at\left(1 - \frac{1}{2}\frac{x^2}{v^2t^2}\right)}}{\sqrt{2\pi at}}.$$

Introducing this value in (43) gives

$$i_x = \frac{E}{Lv}\frac{\epsilon^{-\frac{ax^2}{2v^2t^2}}}{\sqrt{2\pi at}} = \frac{E\epsilon^{-\frac{RCx^2}{4t}}}{\sqrt{\frac{\pi Rt}{C}}}.$$

the same expression that was obtained for this case by direct development (see formula (15), Chap. IV).

DERIVATION OF VOLTAGE WAVE

The formula for the voltage can be obtained from the current formula by the relation between current and voltage, which is

$$-\frac{dV}{dx} = (R + Lp)i,$$

$$= L(p + 2a)i,$$

$$= \frac{E}{v}(p + 2a)\epsilon^{-at}I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\}. \qquad (44)$$

Shifting ϵ^{-at} to the left, changing p to $p - a$, we have

$$-\frac{dV}{dx} = \frac{E}{v}\epsilon^{-at}(p + a)I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\}. \tag{45}$$

The voltage is the negative of the x integral of the right-hand member of the above equation. The integration can be readily effected, using the bracket term of (42), the equivalent of $I_0\left\{\frac{a}{v}(v^2t^2 - x^2)^{1/2}\right\}$, which gives the following:

$$V = -E\epsilon^{-at}\left(1 + \frac{p}{a}\right)\left\{\frac{ax}{v}I_0(at) - \left(\frac{ax}{v}\right)^3\frac{I_1(at)}{2 \cdot 3 \cdot at} + \left(\frac{ax}{v}\right)^5\frac{I_2(at)}{2 \cdot 4 \cdot 5(at)^2} - \cdots\right\} \tag{46}$$

We must add to this a constant of integration to satisfy the condition $V = E$ when $x = 0$. Since the above makes V vanish at the origin, the constant of integration is obviously E. Hence,

$$V = E\epsilon^{-at}\left\{\epsilon^{at} - \left(1 + \frac{p}{a}\right)\left[\left(\frac{ax}{v}\right)I_0(at) - \left(\frac{ax}{v}\right)^3\frac{I_1(at)}{2 \cdot 3at} + \left(\frac{ax}{v}\right)^5\frac{I_2(at)}{2 \cdot 4 \cdot 5(at)^2} - \cdots\right]\right\} \tag{47}$$

The complete development of the solution for V involves one time differentiation. We have, by (40),

$$p\frac{I_m(at)}{(at)^m} = a^2t\frac{I_{m+1}(at)}{(at)^{m+1}},$$

that is,

$$pI_0(at) = aI_1(at),$$

$$p\frac{I_1(at)}{at} = a\frac{I_2(at)}{at},$$

$$p\frac{I_2(at)}{(at)^2} = a\frac{I_3(at)}{(at)^2}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Introducing these values in (47), we obtain the complete developed solution for V as follows:

$$V = E\epsilon^{-at}\left\{\epsilon^{at} - \frac{ax}{v}(I_0 + I_1) + \left(\frac{ax}{v}\right)^3\frac{I_1 + I_2}{2 \cdot 3 \cdot at} - \left(\frac{ax}{v}\right)^5\frac{I_2 + I_3}{2 \cdot 4 \cdot 5(at)^2} + \cdots\right\}. \tag{48}$$

The argument of all the I functions is at.

Another way of deriving the expression for the voltage is to work direct without any regard to the developed current formula from which (48) was derived. The voltage on the line is given by

$$E_x = E\epsilon^{-Kx}$$

$$K = \sqrt{LCp^2 + RCp} = \frac{1}{v}\sqrt{p^2 + 2ap},$$

$$= \frac{1}{v}\sqrt{(p + a)^2 - a^2}.$$

We may put

$$E_x = E\epsilon^{-at}\epsilon^{at}\epsilon^{-\frac{1}{v}\sqrt{(p+a)^2-a^2}x}$$

Shifting ϵ^{at} to the right by changing p to $p - a$, the above transforms to

$$E\epsilon^{-at}\epsilon^{-\frac{1}{v}(p^2-a^2)^{\frac{1}{2}}x}\epsilon^{at}. \tag{49}$$

Expanding the exponential factor $\epsilon^{-\frac{1}{v}(p^2-a^2)^{\frac{1}{2}}x}$ in a series, we obtain

$$E_x = E\epsilon^{-at}\left\{1 - \frac{x}{v}(p^2 - a^2)^{\frac{1}{2}} + \frac{x^2}{2!v^2}(p^2 - a^2) - \frac{x^3}{3!v^3}(p^2 - a^2)^{\frac{3}{2}} + \cdots\right\}\epsilon^{at}, \tag{50}$$

operating by $(p^2 - a^2)$ or an integral power of $(p^2 - a^2)$ on ϵ^{at} is obviously zero; hence, the integral powers of $(p^2 - a^2)$ in the above expression are to be disregarded, and it reduces to

$$E_x = E\epsilon^{-at}\left\{1 - \frac{x}{v}\left(1 + \frac{x^2}{3!v^2}(p^2 - a^2) + \frac{x^4}{5!v^4}(p^2 - a^2)^2 + \cdots\right)(p^2 - a^2)^{\frac{1}{2}}\right\}\epsilon^{at}. \tag{51}$$

By (16),

$$\left(\frac{p - a}{p + a}\right)^{\frac{1}{2}}\epsilon^{at} = \frac{(p^2 - a^2)^{\frac{1}{2}}}{p + a}\epsilon^{at} = I_0(at),$$

and

$$(p^2 - a^2)^{\frac{1}{2}}\epsilon^{at} = (p + a)I_0(at).$$

Therefore,

$$E_x = E\epsilon^{-at}\left\{\epsilon^{at} - \frac{ax}{v}\left(1 + \frac{p}{a}\right)\left(1 + \frac{x^2}{3!v^2}(p^2 - a^2) + \frac{x^4}{5!v^4}(p^2 - a^2)^2 + \cdots\right)I_0(at)\right\}. \tag{52}$$

The results produced by operating by $(p^2 - a^2)$ or integral powers of $(p^2 - a^2)$ on $I_0\,(at)$ are given by (41), and, substituting these values in (52), we obtain

$$E_x = E\epsilon^{-at}\left\{\epsilon^{at} - \frac{ax}{v}\left(1 + \frac{p}{a}\right)\left(I_0(at) - \frac{x^2a^2}{2 \cdot 3 \cdot v^2}\frac{I_1(at)}{at} + \frac{x^4v^4}{2 \cdot 4 \cdot 5v^4}\frac{I_2(at)}{(at)^2} - \cdots\right)\right\} \tag{53}$$

which is the same as (46), and developing it fully will give again, of course, (48).

When all the electrical constants, including leakage, are taken into consideration, and steady voltage applied at the origin, the problem is much more difficult. The final formulas assume a form which is too complex for interpretation, either physically or numerically, and the discussion of this problem is, therefore, omitted here. Heaviside gives the complete derivation beginning on page 312, vol. 2, Electromagnetic Theory.

Applied Voltage Varying as ϵ^{-2bt}.—A special case, however, of some interest is when the applied voltage varies as $\epsilon^{-\frac{g}{C}t} = \epsilon^{-2bt}$. In this case,

$$\begin{aligned} i_x &= \frac{E\epsilon^{-Kx}}{Lv}\sqrt{\frac{p + 2b}{p + 2a}}\epsilon^{-2bt}, \\ &= E\epsilon^{-at}\epsilon^{at}\epsilon^{-\frac{x}{v}\sqrt{(p+2a)(p+2b)}}\sqrt{\frac{p + 2b}{p + 2a}}\epsilon^{-2bt}. \end{aligned}$$

Shift ϵ^{at} to the right and ϵ^{-bt} to the left, which changes p to $p - a - b$, and the above transforms to

$$i_x = \frac{E\epsilon^{-\rho t}}{Lv}\epsilon^{-\frac{x}{v}\sqrt{(p+\sigma)(p-\sigma)}}\sqrt{\frac{p - \sigma}{p + \sigma}}\epsilon^{\sigma t}, \tag{54}$$

By (16), we have

$$\left(\frac{p - \sigma}{p + \sigma}\right)^{\frac{1}{2}}\epsilon^{\sigma t} = I_0(\sigma t).$$

Hence,

$$i_x = \frac{E\epsilon^{-\rho t}}{Lv}\epsilon^{-\frac{x}{v}(p^2-\sigma^2)}I_0(\sigma t). \tag{55}$$

This is exactly the same form as (36), for which the developed solution was obtained, and can be applied to this problem, which gives

$$i_x = \frac{E\epsilon^{-\rho t}}{Lv}I_0\left\{\frac{\sigma}{v}(v^2t^2 - x^2)^{\frac{1}{2}}\right\}. \tag{56}$$

FINITE LINES

The method of treatment adopted in Chap. IV in connection with the solution of problems relating to cables of finite length will be followed here, applying the expansion theorem, which is simpler than the direct operational process. We shall first consider the case of free lines, that is, either grounded or open at the ends, no terminal impedances, in which complete reflections at the ends with or without reversals are produced. The investigation will be carried through for two cases: steady and alternating voltage applied at the origin.

Line Open at Far End, and Steady Voltage Applied at Origin.—The voltage and current at any point on the line are given by (6) and (8),

$$\left.\begin{aligned} V &= A\epsilon^{Kx} + B\epsilon^{-Kx}, \\ i &= \frac{1}{Lv}\left(\frac{p+2b}{p+2a}\right)^{1/2}\{-A\epsilon^{Kx} + B\epsilon^{-Kx}\}. \\ K &= \frac{1}{v}(p+2a)^{1/2}(p+2b)^{1/2}. \end{aligned}\right. \tag{57}$$

The terminal conditions for this problem are

$$x = 0;\ V = E,$$
$$x = l;\ i = 0.$$

Hence,

$$A + B = E$$
$$-A\epsilon^{Kl} + B\epsilon^{-Kl} = 0.$$

From which the values of A and B are determined as follows:

$$\left.\begin{aligned} A &= \frac{E\epsilon^{-Kl}}{\epsilon^{Kl} + \epsilon^{-Kl}}, \\ B &= \frac{E\epsilon^{Kl}}{\epsilon^{Kl} + \epsilon^{-Kl}}. \end{aligned}\right\} \tag{58}$$

Introducing these values in (57) gives

$$\left.\begin{aligned} V &= \frac{E\{\epsilon^{K(l-x)} + \epsilon^{-K(l-x)}\}}{\epsilon^{Kl} + \epsilon^{-Kl}} \\ i &= \frac{E}{Lv}\left(\frac{p+2b}{p+2a}\right)^{1/2}\left\{\frac{\epsilon^{K(l-x)} - \epsilon^{-K(l-x)}}{\epsilon^{Kl} + \epsilon^{-Kl}}\right\}. \end{aligned}\right. \tag{59}$$

Expressed in hyperbolic functions,

$$V = \frac{E \cosh K(l - x)}{\cosh Kl}$$
$$i = \frac{E}{Lv}\left(\frac{p + 2b}{p + 2a}\right)^{\frac{1}{2}} \frac{\sinh K(l - x)}{\cosh Kl}. \tag{60}$$

The expansion theorem can be applied to either V or i; but it is sufficient to develop the solution for one, the other is obtained from the circuital relation between V and i. We shall develop the solution for V.

The determinantal equation is

$$Z(p) = \cosh Kl = 0. \tag{61}$$

Hence,

$$Kl = j\frac{n\pi}{2};\ n = 1, 3, 5 \cdots,$$

and

$$v^2K^2 = (p + 2a)(p + 2b) = -\frac{n^2\pi^2v^2}{4l^2}.$$

From this, the values of p are readily determined.

$$\left.\begin{aligned} p_n &= -\rho \pm \sqrt{\sigma^2 - \frac{n^2\pi^2v^2}{4l^2}}, \\ &= -\rho \pm j\beta_n, \\ \beta_n &= \sqrt{\frac{n^2\pi^2v^2}{4l^2} - \sigma^2}. \end{aligned}\right\} \tag{62}$$

Also,

$$\frac{\partial Z(p)}{\partial p} = \sinh Kl\frac{\partial (Kl)}{\partial p}$$
$$= \frac{\sinh Kl}{Kl}(p + \rho)\frac{l^2}{v^2}. \tag{63}$$

Introducing the value of $Kl = jn\pi/2$, and the value of p as given by (62), the above transforms to

$$\frac{\partial Z(p)}{\partial p} = \pm j\frac{\sin\frac{n\pi}{2}\cdot j\beta_n\frac{l^2}{v^2}}{j\frac{n\pi}{2}} = \pm 2j\frac{l^2}{v^2}\beta_n\frac{\sin\frac{n\pi}{2}}{n\pi}. \tag{64}$$

For $p = 0$,

$$Z(p)_{p=0} = \frac{\cosh \frac{2\sqrt{ab}}{v} l}{\cosh \frac{2\sqrt{ab}}{v}(l - x)} = \frac{\cosh sl}{\cosh s(l - x)}. \tag{65}$$

where, for brevity, we put $s = 2\sqrt{ab}/v$.

Substituting in the expansion-theorem formula, we obtain the developed expression for the voltage,

$$V = \frac{E \cosh s(l - x)}{\cosh sl} + E \sum \frac{\epsilon^{-\rho t} \epsilon^{\pm j\beta_n t} \cos \frac{n\pi}{2l}(l - x)}{\pm(-\rho \pm j\beta_n) \frac{2jl^2}{n\pi v^2} \beta_n \sin \frac{n\pi}{2}}$$

$$= \frac{E \cosh s(l - x)}{\cosh sl} + E\epsilon^{-\rho t} \sum \frac{\epsilon^{\pm j\beta_n t} n\pi v^2 \sin \frac{n\pi x}{2l}}{\pm 2jl^2 \beta_n(-\rho \pm j\beta_n)}. \tag{66}$$

The complete solution must include all terms which come under the double sign in the summation term. Hence,

$$V = \frac{E \cosh s(l - x)}{\cosh sl} + \frac{E\pi v^2}{2l^2} \epsilon^{-\rho t} \sum n \sin \frac{n\pi x}{2l}$$
$$\left\{ \frac{\cos \beta_n t + j \sin \beta_n t}{j\beta_n(-\rho + j\beta_n)} + \frac{\cos \beta_n t - j \sin \beta_n t}{j\beta_n(\rho + j\beta_n)} \right\}.$$

which simplifies to

$$V = \frac{E \cosh s(l - x)}{\cosh sl} - E\frac{\pi v^2}{l^2} \epsilon^{-\rho t} \sum$$
$$\frac{n \sin \frac{n\pi x}{2l}(\rho \sin \beta_n t + \beta_n \cos \beta_n t)}{\beta_n(\rho^2 + \beta_n)}. \tag{67}$$

This is the complete solution for the voltage, the first right-hand term giving the steady-state condition, which theoretically obtains only after an infinite time, and the summation term giving the transient voltage component, the summation to be extended for all odd integrals from 1 to ∞. For $x = 0$, the summation term is zero, and the first right-hand term reduces to E, as it should.

The expression for the current can be derived from (67) by the circuital relation

$$-\frac{di}{dx} = (Cp + g)V.$$

This gives

$$-\frac{di}{dx} = Eg\frac{\cosh s\,(l - x)}{\cosh sl} - E\frac{\pi v^2}{l^2}\sum n \sin\frac{n\pi x}{2l}$$
$$\left\{\frac{g\rho \sin \beta_n t + g\beta_n \cos \beta_n t}{\beta_n(\rho^2 + \beta_n^2)} - C\frac{\sin \beta_n t}{\beta_n}\right\},$$

and

$$-i = \frac{Eg}{s}\frac{\sinh s(l - x)}{\cosh sl} + E\frac{2v^2}{l}\epsilon^{-\rho t}\sum \cos\frac{n\pi x}{2l}$$
$$\left\{\frac{g\rho \sin \beta_n t + g\beta_n \cos \beta_n t}{\beta_n(\rho^2 + \beta_n^2)} - C\frac{\sin \beta_n t}{\beta_n}\right\}, \quad (68)$$

which is the complete developed expression for the current; the first right-hand term is the steady-state component, and the summation term, the transient component.

For $g = 0$, negligible leakage, equation (66) reduces to ($p = a$, in this case)

$$i = \frac{E2v^2}{l}C\epsilon^{-at}\sum\frac{\cos\frac{n\pi x}{2l}\sin \beta_n t}{\beta_n}. \qquad (69)$$

Line Grounded at the Far End.—The terminal conditions for this case are

$$x = 0;\ V = E,$$
$$x = l;\ V = 0,$$

which give, on substitution in (57),

$$A + B = E,$$
$$A\epsilon^{Kl} + B\epsilon^{-Kl} = 0.$$

From these, A and B are determined,

$$A = \frac{-E\epsilon^{-Kl}}{\epsilon^{Kl} - \epsilon^{-Kl}},$$
$$B = \frac{E\epsilon^{Kl}}{\epsilon^{Kl} - \epsilon^{-Kl}}.$$

Substituting these values in (57), we obtain the following expressions for the voltage and current:

$$\left.\begin{aligned} V &= E\left\{\frac{\epsilon^{K(l-x)} - \epsilon^{-K(l-x)}}{\epsilon^{Kl} - \epsilon^{-Kl}}\right\}, \\ i &= \frac{E}{Lv}\left(\frac{p+2b}{p+2a}\right)^{\frac{1}{2}}\left\{\frac{\epsilon^{K(l-x)} + \epsilon^{-K(l-x)}}{\epsilon^{Kl} - \epsilon^{-Kl}}\right\}. \end{aligned}\right\} \tag{70}$$

Expressed in hyperbolic functions,

$$\left.\begin{aligned} V &= \frac{E \sinh K(l-x)}{\sinh Kl}, \\ i &= \frac{E}{Lv}\left(\frac{p+2b}{p+2a}\right)^{\frac{1}{2}}\frac{\cosh K(l-x)}{\sinh Kl}. \end{aligned}\right\} \tag{71}$$

Applying the expansion theorem for the development of the solution from the above expressions, as in the preceding case, we have here the determinant

$$Z(p) = \sinh Kl = 0. \tag{72}$$

Hence,

$$Kl = jn\pi,$$

and

$$K = \frac{1}{v}\sqrt{(p+2a)(p+2b)} = j\frac{n\pi}{l},$$

from which p is determined as follows:

$$\left.\begin{aligned} p_n &= -\rho \pm \sqrt{\sigma^2 - \frac{n^2\pi^2v^2}{l^2}}, \\ &= -\rho \pm j\beta_n \\ \beta_n &= \sqrt{\frac{n^2\pi^2v^2}{l^2} - \sigma^2}. \end{aligned}\right\} \tag{73}$$

Also,

$$\left.\begin{aligned} \frac{\partial Z(p)}{\partial p} &= \cosh Kl\frac{\partial(Kl)}{\partial p}, \\ &= \frac{l^2}{v^2}(p+\rho)\frac{\cosh Kl}{Kl} \end{aligned}\right\} \tag{74}$$

For $p = 0$,

$$Z(p)_{p=0} = \frac{\sinh \dfrac{2\sqrt{ab}l}{v}}{\sinh \dfrac{2\sqrt{ab}}{v}(l-x)} = \frac{\sinh sl}{\sinh s(l-x)}. \tag{75}$$

Substituting these values from (73), (74), and (75), in the expansion formula, we obtain the developed solution for the voltage, as follows:

$$V = \frac{E \sinh s(l - x)}{\sinh sl} + E \sum \frac{j \sin n\pi\left(1 - \frac{x}{l}\right)\epsilon^{-\rho t}\epsilon^{\pm j\beta_n t}}{\pm j\beta_n(-\rho \pm j\beta_n)\frac{l^2}{v^2}\cos(n\pi)\frac{1}{jn\pi}}$$

$$= \frac{E \sinh s(l - x)}{\sinh sl} - E\frac{v^2}{l^2}\pi\epsilon^{-\rho t}\sum \frac{n \sin \frac{n\pi x}{l}\epsilon^{\pm j\beta_n t}}{\pm j\beta_n(-\rho \pm j\beta_n)} \qquad (76)$$

The complete solution to include both terms which come under the double sign in the summation term is

$$V = \frac{E \sinh s(l - x)}{\sinh sl} - \frac{E\pi v^2}{l^2}\epsilon^{-\rho t}\sum n \sin \frac{n\pi x}{l}\left\{\frac{\cos \beta_n t + j \sin \beta_n t}{j\beta_n(-\rho + j\beta_n)} + \frac{\cos \beta_n t - j \sin \beta_n t}{j\beta_n(\rho + j\beta_n)}\right\},$$

which simplifies to

$$V = \frac{E \sinh s(l - x)}{\sinh sl} - \frac{2Ev^2\pi}{l^2}\epsilon^{-\rho t} \sum \frac{n \sin \frac{n\pi x}{l}\{\rho \sin \beta_n t + \beta_n \cos \beta_n t\}}{\beta_n(\rho^2 + \beta_n^2)}. \qquad (77)$$

This is the complete solution, the first term giving the steady-state voltage, and the summation term, the transient state, the summation to extend for all integral values of n from 0 to ∞. When $x = 0$, the above reduces to $V = E$, the applied voltage at the origin.

The expression for the current is obtained from the voltage formula by the circuital relation

$$-\frac{di}{dx} = (g + Cp)V,$$

which gives

$$-\frac{di}{dx} = Eg\frac{\sinh s(l - x)}{\sinh sl} - \frac{2Ev^2}{l^2}\pi\epsilon^{-\rho t}\sum n \sin \frac{n\pi x}{l}\left\{\frac{g\rho \sin \beta_n t + g\beta_n \cos \beta_n t}{\beta_n(\rho^2 + \beta_n^2)} - C\frac{\sin \beta_n t}{\beta_n}\right\},$$

and

$$i = Eg\frac{\cosh s(l - x)}{\sinh sl} - \frac{2Ev^2}{l}\epsilon^{-\rho t}\sum \cos\frac{n\pi x}{l}$$
$$\left\{\frac{g\rho \sin \beta_n t + g\beta_n \cos \beta_n t}{\beta_n(\rho^2 + \beta_n^2)} - C\frac{\sin \beta_n t}{\beta_n}\right\}. \quad (78)$$

For $g = 0$, (78) and (79) reduce to the following:

$$V = E\left(1 - \frac{x}{l}\right) - \frac{2E\pi v^2}{l^2}\epsilon^{-at}$$
$$\sum \frac{n \sin \frac{n\pi x}{l}\{a \sin \beta_n t + \beta_n \cos \beta_n t\}}{\beta_n(a^2 + \beta_n^2)}. \quad (79)$$

$$i = \frac{2Ev^2}{l}C\epsilon^{-at}\sum \cos\frac{n\pi x}{l}\frac{\sin \beta_n t}{\beta_n}. \quad (80)$$

If the line is infinitely long, either (69) or (80) should give the same result as (43), which was obtained by direct operational process for the same condition. The equivalence of the formulas (43) and (80) for $l = \infty$ is not apparent, but it can be shown that (80) transforms into (43), which we shall now do. The summation term goes by steps of π/l, which become infinitely small when l is made infinitely great. We may put, therefore,

$$\frac{n\pi}{l} = m; \frac{\pi}{l} = dm; \beta_m = \sqrt{\frac{n^2\pi^2v^2}{l^2} - a^2} = \sqrt{m^2v^2 - a^2}, \quad (81)$$

and the summation term in (80) is converted into a definite integral, that is,

$$i = \frac{2Ev^2C}{\pi}\epsilon^{-at}\int_0^\infty \cos mx \frac{\sin \beta_m t}{\beta_m}dm. \quad (82)$$

Evaluation of the integral,

$$u = \int_0^\infty \cos mx \frac{\sin \beta_m t}{\beta_m}dm, \quad (83)$$
$$\beta_m = \sqrt{m^2v^2 - a^2}.$$

β is a function of a^2; differentiating $\sin \beta t/\beta$ with respect to t and a^2, we obtain

$$\frac{d^2}{dtda^2}\frac{\sin \beta t}{\beta} = \frac{d}{da^2}\cos \beta t = \frac{1}{2}t\frac{\sin \beta t}{\beta}, \quad (84)$$

that is,

$$\frac{d}{da^2}\frac{\sin \beta t}{\beta} = \frac{1}{2}\int_0^t \frac{t \sin \beta t}{\beta} dt = \frac{t}{2p}\frac{\sin \beta t}{\beta}. \tag{85}$$

By Taylor's theorem,

$$f(a^2) = f(0) + a^2 f'(0) + \frac{a^4}{2!}f''(0) + \cdots$$

Considering $\sin \beta t/\beta$ as a function of a^2, and remembering that for $a = 0$, $\beta = mv$, we get, by Taylor's theorem, replacing the differentiations by successive operations of $t/2p$ in accordance with (85), the following:

$$\frac{\sin \beta t}{\beta} = \frac{\sin mvt}{mv} + \frac{a^2 t}{2p}\frac{\sin mvt}{mv} + \left(\frac{a^2 t}{2p}\right)^2 \frac{\sin mvt}{2!mv} + \cdots \tag{86}$$

Substituting this in (83), we get

$$u = \left\{ 1 + \frac{a^2 t}{2p} + \frac{\left(\frac{a^2 t}{2p}\right)^2}{2!} + \cdots \right\} \int_0^\infty \cos mx \frac{\sin mvt}{mv} dm. \tag{87}$$

The integral may be put in this form:

$$\int_0^\infty \frac{\cos mx \sin mvt}{mv} dm = \frac{1}{2}\int_0^\infty \frac{\{\sin m(vt + x) + \sin m(vt - x)\}}{mv} dm.$$

We know, however, that

$$\int_0^\infty \frac{\sin mx}{m} dm = \pm\frac{\pi}{2}, \tag{88}$$

having positive value for x positive and a negative value for x negative. Hence,

$$\begin{aligned}\int_0^\infty \frac{\cos mx \sin mvt}{mv} dm &= \frac{\pi}{4}(1 + 1) = \frac{\pi}{2} \text{ when } x < vt,\\ &= \frac{\pi}{4}(1 - 1) = 0 \text{ when } x > vt,\end{aligned} \tag{89}$$

that is, the operand which is the integral in (87) is zero when $t < \frac{x}{v}$.

If we use the operand 1 only in (87), the limits of integration must be from x/v to t instead of 0 to t. We finally arrive at the following:

$$u = \frac{\pi}{2v}\left\{1 + \frac{1}{2}a^2\int_{\frac{x}{v}}^{t} tdt + \frac{(\frac{1}{2}a^2)^2}{2!}\int_{\frac{x}{v}}^{t} tdt\int_{\frac{x}{v}}^{t} tdt + \cdots\right\}$$

$$= \frac{\pi}{2v}\left\{1 + \frac{1}{2}\frac{a^2}{2}\left(t^2 - \frac{x^2}{v^2}\right) + \frac{1}{8}\frac{(\frac{1}{2}a^2)^2}{2!}\left(t^2 - \frac{x^2}{v^2}\right)^2 + \frac{1}{48}\frac{(\frac{1}{2}a^2)^3}{3!}\left(t^2 - \frac{x^2}{v^2}\right)^3 + \cdots\right\} \quad (90)$$

If we put $a\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}} = y$, the above takes the form

$$u = \frac{\pi}{2v}\left\{1 + \frac{y^2}{2^2} + \frac{y^4}{2^2 \cdot 4^2} + \frac{y^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right\}$$

$$= \frac{\pi}{2v}I_0(y) = \frac{\pi}{2v}I_0\left\{a\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right\}. \quad (91)$$

Substituting the value of u from (91), which is the value of the definite integral, in (82), we arrive at the expression for the current,

$$i = \frac{2Ev^2C\epsilon^{-at}}{\pi}\frac{\pi}{2v}I_0\left\{a\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right\}$$

$$= \frac{E\epsilon^{-at}}{Lv}I_0\left\{a\left(t^2 - \frac{x^2}{v^2}\right)^{\frac{1}{2}}\right\}, \quad (92)$$

which is the same as (43), and thus establish the equivalence of the result obtained by two different processes—the direct operational process and the application of the expansion theorem.

Terminal Impedances.—If the effects of the terminal apparatus at either end of the line are to be included in the general solution, the problem is by far more difficult. It is only in special cases that it is at all possible to obtain the completely developed solution by the application of the expansion theorem. We shall give here only an outline of the problem and indicate some special cases where a solution is possible.

Consider the most general case: a line of length l, impedances Z_t and Z_r connected at the transmitting and receiving ends, respectively. The solutions of the differential equations given

by (6) and (7) are perfectly general and also applicable for this case. We have

$$\left.\begin{aligned} V &= A\epsilon^{Kx} + B\epsilon^{-Kx}, \\ i &= \frac{K}{Lp + R}\{-A\epsilon^{Kx} + B\epsilon^{-Kx}\}, \end{aligned}\right\} \tag{93}$$

where

$$K^2 = (Lp + R)(Cp + g). \tag{94}$$

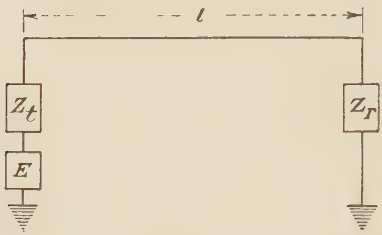

FIG. 22.

The constants A and B are to be determined from the terminal conditions, which, in this case, are
when

$$\left.\begin{aligned} x = 0;\ V_0 &= E - Z_t i_0. \\ x = l;\ V_l &= Z_r i_l. \end{aligned}\right\} \tag{95}$$

Introducing the values of $V_{x=0}$ and $i_{x=0}$ from (93) into (95), we get the following two equations:

$$A + B = E - \frac{Z_t K}{Lp + R}(-A + B),$$

$$A\epsilon^{Kl} + B\epsilon^{-Kl} = Z_r \frac{K}{Lp + R}(-A\epsilon^{Kl} + B\epsilon^{-Kl}).$$

Rearranging, we get

$$\left.\begin{aligned} \left(1 - \frac{Z_t K}{Lp + R}\right)A + \left(1 + \frac{Z_t K}{Lp + R}\right)B &= E, \\ \left(1 + \frac{Z_r K}{Lp + R}\right)A\epsilon^{Kl} + \left(1 - \frac{Z_r K}{Lp + R}\right)B\epsilon^{-Kl} &= 0. \end{aligned}\right\} \tag{96}$$

Solving for A and B gives the following:

$$A = \frac{-E\left(1 - \frac{Z_r K}{Lp + R}\right)\epsilon^{-Kl}}{\left(1 + \frac{Z_r Z_t K^2}{(Lp+R)^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + (Z_r + Z_t)\frac{K}{Lp + R}(\epsilon^{Kl} + \epsilon^{-Kl})},$$

$$B = \frac{E\left(1 + \frac{Z_r K}{Lp + R}\right)\epsilon^{Kl}}{\left(1 + \frac{Z_r Z_t K^2}{(Lp + R)^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + (Z_r + Z_t)\frac{K}{Lp + R}(\epsilon^{Kl} + \epsilon^{-Kl})}. \tag{97}$$

Introducing these values into (93), we obtain the expressions for V and i as follows:

$$V = \frac{E\left\{\left(1 + \frac{Z_r K}{Lp + R}\right)\epsilon^{K(l-x)} - \left(1 - \frac{Z_r K}{Lp + R}\right)\epsilon^{-K(l-x)}\right\}}{\left(1+\frac{Z_r Z_t K^2}{(Lp+R)^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + (Z_r + Z_t)\frac{K}{Lp+R}(\epsilon^{Kl} + \epsilon^{-Kl})},$$

$$i = \frac{E\frac{K}{Lp+R}\left\{\left(1+\frac{Z_r K}{Lp+R}\right)\epsilon^{K(l-x)} + \left(1-\frac{Z_r K}{Lp+R}\right)\epsilon^{-K(l-x)}\right\}}{\left(1+\frac{Z_r Z_t K^2}{(Lp+R)^2}\right)(\epsilon^{Kl} - \epsilon^{-Kl}) + (Z_r + Z_t)\frac{K}{Lp+R}(\epsilon^{Kl} + \epsilon^{-Kl})}. \tag{98}$$

Expressed in hyperbolic functions as follows:

$$V = \frac{E\left\{\sinh K(l - x) + \frac{Z_r K}{Lp + R}\cosh K(l - x)\right\}}{\left(1 + \frac{Z_r Z_t K^2}{(Lp + R)^2}\right)\sinh Kl + \frac{K}{Lp + R}(Z_r + Z_t)\cosh Kl},$$

$$i = \frac{\frac{EK}{Lp + R}\left\{\cosh K(l - x) + \frac{Z_r K}{Lp + R}\sinh K(l - x)\right\}}{\left(1 + \frac{Z_r Z_t K^2}{(Lp + R)^2}\right)\sinh Kl + \frac{K}{Lp + R}(Z_r + Z_t)\cosh Kl}. \tag{99}$$

To develop the complete solution from the above equations will necessitate obtaining the roots of the determinantal equation

$$Z(p) = \frac{Lp + R}{K}\left\{\left(1 + \frac{Z_r Z_t K^2}{(Lp + R)^2}\right)\sinh Kl + \frac{K}{Lp + R}(Z_r + Z_t)\cosh Kl\right\} = 0, \tag{100}$$

which for most combinations of Z_t and Z_r is nearly impossible to do. In some special cases, however, the roots of this equation can be determined, which makes it then possible to apply the expansion formula and develop the solution.

The determinantal equation may be put in a more convenient form. We have, for $x = 0$,

$$i = \frac{E\dfrac{K}{Lp+R}\left\{\cosh Kl + \dfrac{Z_rK}{Lp+R}\sinh Kl\right\}}{\left(1 + \dfrac{Z_rZ_tK^2}{(Lp+R)^2}\right)\sinh Kl + \dfrac{K}{Lp+R}(Z_t + Z_r)\cosh Kl},$$

$$= \frac{E}{Z(p)}. \tag{101}$$

where

$$Z(p) = \frac{\left(1 + \dfrac{Z_rZ_tK^2}{(Lp+R)^2}\right)\tanh Kl + \dfrac{K}{Lp+R}(Z_r + Z_t)}{\dfrac{K}{Lp+R}\left\{1 + \dfrac{Z_rK}{Lp+R}\tanh Kl\right\}},$$

$$= \frac{\left(\dfrac{Lp+R}{K} + \dfrac{Z_rZ_tK}{Lp+R}\right)\tanh Kl + Z_r + Z_t}{1 + \dfrac{Z_rK}{Lp+R}\tanh Kl} \tag{102}$$

The above may be put in this form:

$$Z(p) = Z_t + \frac{\dfrac{R+Lp}{K}\tanh Kl + Z_r}{1 + \dfrac{KZ_r}{R+Lp}\tanh Kl} = 0. \tag{103}$$

Consider now a special case: $Z_t = 0$, and $Z_r = nl(Lp + R)$; n is a numerical factor, the receiving apparatus consisting of a coil whose time constant is L/R. For this condition, (103) reduces to the following:

$$Z(p) = \frac{(R+Lp)\left(\dfrac{1}{K}\tanh Kl + nl\right)}{1 + nKl\tanh Kl} = 0,$$

$$= \frac{(R+Lp)l\left(\dfrac{\tanh Kl}{Kl} + n\right)}{1 + nKl\tanh Kl} = 0. \tag{104}$$

The roots of the equation $Z(p) = 0$, (104), are given by

$$\left.\begin{aligned} R + Lp &= 0, \\ \tanh Kl + nKl &= 0. \end{aligned}\right\} \tag{105}$$

For the first equation (105), we have the solitary root

$$p = -\frac{R}{L}. \tag{106}$$

The roots of the second equation (105) can be obtained graphically. Put

$$\begin{aligned} Kl &= jx. \\ nKl &= jnx, \\ \tanh Kl &= \tanh jx = -j \tan x, \end{aligned}$$

and the second equation (105) transforms to

$$nx = \tan x. \tag{107}$$

The intersecting points of the curves $y_1 = nx$ and $y_2 = \tan x$ give the values of x which are the roots of (107). From the relation $Kl = jx$, the values of p corresponding to root values of x are readily obtained.

To complete the solution, it remains yet to obtain expressions for the $p\dfrac{\partial Z}{\partial p}$ terms of the expansion formula.

For the root $p = -R/L$, we note that for this value of p, $K = 0$. Hence,

$$\tanh Kl = 0,$$

and

$$\frac{\tanh Kl}{kl} = 1,$$

which gives

$$Z(p) = (R + Lp)l(1 + n).$$

Therefore,

$$p\frac{\partial Z(p)}{\partial p} = -\frac{R}{L}Ll(1 + n) = Rl(1 + n). \tag{108}$$

For the other roots, we have

$$\begin{aligned} \frac{\partial Z(p)}{\partial p} &= (R + Lp)l\left\{\frac{Kl^2}{\cosh^2 Kl} + nKl^2\right\}\frac{\partial K}{\partial p}, \\ &= \frac{(R + Lp)Kl^3}{\cosh^2 Kl}\{1 + n \cosh^2 Kl\}\frac{\partial K}{\partial p}. \end{aligned} \tag{109}$$

This gives all the data necessary from which the complete solution can be realized by the application of the expansion formula.

It must be remembered, however, that to get the solution of either the current or voltage at any point distance x from the transmitter, each of the summation terms in the expansion formula are to be multiplied by the numerators of either equation (101), using the proper root values of p.

Another case which permits of a similar reduction is the following:

$$\left.\begin{aligned} Z_t &= n_1(R + Lp) + n_2(g + Cp)^{-1}, \\ Z_r &= n_3(R + Lp) + n_4(g + Cp)^{-1}. \end{aligned}\right\} \qquad (110)$$

n_1, n_2, n_3, n_4 being numerics. That is, apparatus at either end consisting of a coil and a condenser in series, the time constant of the coil being L/R and that of the condenser C/g.

By (103), we may put the determinantal equation in this form:

$$Z(p) = \frac{Z_t + \dfrac{Z_tZ_rK}{Lp + R}\tanh Kl + \dfrac{R + Lp}{K}\tanh Kl + Z_r}{1 + \dfrac{Z_rK}{Lp + R}\tanh Kl} = 0. \quad (111)$$

This gives the relation

$$\frac{\tanh Kl}{Kl} = \frac{-(Z_t + Z_r)}{Z_rZ_t(g + Cp)l + (R + Lp)l}. \qquad (112)$$

Substituting the values of Z_t and Z_r from (111), we get

$$\frac{\tanh Kl}{Kl} = \frac{-(n_1 + n_3)(R + Lp) - (n_2 + n_4)(g + Cp)^{-1}}{(g + Cp)l\left\{n_1n_3(R + Lp)^2 + (n_1n_4 + n_2n_3)\left(\dfrac{R + Lp}{g + Cp}\right) + \dfrac{n_2n_4}{(g + Cp)^2}\right\} + (R + Lp)l.}$$

$$= \frac{-(n_1 + n_3)K^2 - (n_2 + n_4)}{l\{n_1n_3K^4 + (n_1n_4 + n_2n_3)K^2 + n_2n_4\} + lK^2}, \qquad (113)$$

$$\tanh Kl = \frac{-\{(n_1 + n_3)K^3 + (n_2 + n_4)K\}}{n_1n_3K^4 + (n_1n_4 + n_2n_3 + 1)K^2 + n_2n_4}. \qquad (114)$$

The roots of this equation are readily determined by graphic method, plotting the curves

$$\left.\begin{aligned} y_1 &= \tanh Kl, \\ y_2 &= f(Kl). \end{aligned}\right\} \qquad (115)$$

$f(Kl)$ is the right-hand side of equation (114), the intersecting points of the two sets of curves giving the roots of the equation. Once it is possible to get the values of the roots of the determinantal equation, the solution can be completely developed by the application of the expansion formula.

CHAPTER VI

ARTIFICIAL LINES

Artificial lines are of practical importance in their use for balancing telephone lines or ocean telegraph cables for duplexing or other purposes. They also find application in the study of electrical phenomena connected with transmission problems on long lines. An artificial line closely simulating the characteristics of an actual line permits experimental laboratory study of transmission-line phenomena which would be otherwise impractical. It is, therefore, of some importance to investigate the problem of wave propagation on artificial lines and particularly in the matter of transients. It will also serve our purpose in further illustrating the utility of the operational method and the expansion theorem for the solution of a difficult and important problem.

An artificial line is essentially a circuit structure consisting of a large number of recurrent sections of series and shunt impedances, the series impedance of each section representing the resistance or resistance and inductive reactance of unit length of line, while the shunt impedance represents the capacity reactance, or the capacity and leakage combined, of unit length of line. In structure, it is similar to the filter circuit discussed in Chap. III, and the basic formulas (4) and (11) derived there apply also to the artificial-line problem. The fundamental difference is in the number of sections. For an artificial line to simulate closely an actual line, theoretically the number of sections should be infinite in number, or, at least, a very large number. In the filter circuit, only a few sections are sufficient for many purposes.

If the total number of sections is n, the last section being the nth section and the first section designated as the zero section, then for any intermediate section, say the qth section, we have, by (11) and (4), Chap. III,

$$\dot{i}_q = \frac{E \cosh (n - q)\gamma}{z_2 \sinh \gamma \sinh n\gamma,} \tag{1}$$

and

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2}. \tag{2}$$

These are the general formulas applicable for any type of artificial line. What we are generally concerned with most is to obtain an expression for either the input current or the output current, that is, i_0 and i_n.

Artificial Ocean Cable.—Consider the case of an artificial ocean telegraph cable, negligible inductance and leakage, $L = 0$, $g = 0$. In this case,

$$\left.\begin{aligned} z_1 &= R, \\ z_2 &= \frac{1}{Cp}. \end{aligned}\right\} \tag{3}$$

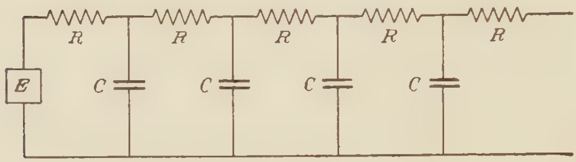

Fig. 23

If we assume an infinite number of sections, $n = \infty$, then $\cosh n\gamma = \sinh n\gamma$, and (1) reduces to, for the input current,

$$i_0 = \frac{E}{z_2 \sinh \gamma}. \tag{4}$$

By (2),

$$\sinh \gamma = \sqrt{\cosh^2 \gamma - 1} = \sqrt{\left(1 + \frac{1}{2}\frac{z_1}{z_2}\right)^2 - 1}$$

$$= \sqrt{\frac{z_1}{z_2} + \frac{1}{4}\frac{z_1^2}{z_2^2}},$$

and

$$z_2 \sinh \gamma = \sqrt{z_1 z_2 + \tfrac{1}{4} z_1^2}. \tag{5}$$

Introducing the values of z_1 and z_2, we get

$$z_2 \sinh \gamma = \sqrt{\frac{R}{Cp} + \frac{1}{4}R^2}. \tag{6}$$

Hence,

$$i_0 = \frac{E}{\sqrt{\dfrac{R}{Cp} + \dfrac{1}{4}R^2}} = \frac{E}{\dfrac{1}{2}R\sqrt{1 + \dfrac{4}{RCp}}}. \tag{7}$$

The complete developed solution is readily derived from (7) by direct operational process. Put, for brevity, $h = 4/Rc$ and expand the denominator in a series of inverse powers of p, thus:

$$\frac{1}{\sqrt{1+\frac{4}{RCp}}} = \frac{1}{\sqrt{1+\frac{h}{p}}} = 1 - \frac{1}{2}\frac{h}{p} + \frac{1\cdot 3}{2\cdot 4}\frac{h^2}{p^2} - \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}\frac{h^3}{p^3} + \cdots$$

Remembering that operating on unity function gives $1/p_n = t^n/n!$, we have, by substitution,

$$\frac{1}{\sqrt{1+\frac{h}{p}}} = 1 - \frac{1}{2}ht + \frac{1\cdot 3}{2\cdot 4}\frac{h^2t^2}{2!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{h^3t^3}{3!} + \cdots,$$

which may be put in this form:

$$\frac{1}{\sqrt{1+\frac{h}{p}}} = 1 - \left(\frac{ht}{2}\right) + \frac{1\cdot 3}{(2!)^2}\left(\frac{ht}{2}\right)^2 - \frac{1\cdot 3\cdot 5}{(3!)^2}\left(\frac{ht}{2}\right)^3 + \cdots \quad (8)$$

The series of (8) is the expanded form of the function $\epsilon^{-h/2\,t}I_0(ht/2)$, which can be readily verified by using the series for the two factors, multiplying and combining terms of the same powers of $ht/2$, and comparing with the corresponding terms in (8). Hence,

$$\frac{1}{\sqrt{1+\frac{h}{p}}} = \epsilon^{-\frac{h}{2}t}I_0\left(\frac{ht}{2}\right). \quad (9)$$

Introducing this value in (7) gives the complete solution for the current in the first section,

$$\begin{aligned} i_0 &= \frac{2E}{R}\epsilon^{-\frac{h}{2}t}I_0\left(\frac{ht}{2}\right), \\ &= \frac{2E}{R}\epsilon^{-\frac{2t}{RC}}I_0\left(\frac{2t}{RC}\right). \end{aligned} \quad (10)$$

If the number of sections is large, the resistance and capacity per section small, the product RC is a very small quantity. For values of t, therefore, differing appreciably from zero, the argu-

ment $2t/RC$ of the Bessel function in (10) is very large, and, to a high degree of approximation,[1]

$$I_0\left(\frac{2t}{RC}\right) = \frac{\epsilon^{\frac{2t}{RC}}}{\sqrt{\frac{2\pi 2t}{RC}}}. \tag{11}$$

Substituting the value from (11) in (10), it reduces to the following:

$$i_0 = \frac{2E}{R}\frac{1}{\sqrt{\frac{4\pi t}{RC}}} = E\sqrt{\frac{C}{R}}\frac{1}{\sqrt{\pi t}}. \tag{12}$$

This is precisely the same as formula (9), Chap. IV, derived for the input current in a cable of uniform distributed resistance and capacity. This condition of the identity of the character of the input currents in a uniform cable and an artificial cable is realized only theoretically when the number of sections in the artificial cable is infinitely large. Practically, this condition is approximated with a reasonably large number of sections.

To obtain the solution for the output current, that is, the current in the nth section, it is best to apply the expansion theorem.

For $q = n$, (1) reduces to

$$i_n = \frac{E}{z_2 \sinh \gamma \sinh n\gamma}. \tag{13}$$

The determinantal equation is

$$Z(p) = z_2 \sinh \sinh n\gamma = 0, \tag{14}$$

which gives

$$n\gamma = js\pi, \text{ and } \gamma = j\frac{s\pi}{n}. \quad (s = 1 \cdot 2 \cdot 3 \cdot \cdot \cdot \tag{15}$$

By 2,

$$\cosh \gamma = \cos \frac{s\pi}{n} = 1 + \frac{1}{2}RCp,$$

which determines the values of p, thus:

$$p = \frac{2}{RC}\left(\cos \frac{s\pi}{n} - 1\right)$$

$$= -\frac{4 \sin^2 \frac{s\pi}{2n}}{RC}. \tag{16}$$

[1] See note on Bessel functions.

The expression for $\frac{\partial Z(p)}{\partial p}$ is readily obtained,

$$\frac{\partial Z(p)}{\partial p} = n \sinh \gamma \cosh n\gamma \frac{\partial \gamma}{\partial p}. \tag{17}$$

By (15), however, we have

$$\sinh \gamma \frac{\partial \gamma}{\partial p} = \frac{1}{2}RC. \tag{18}$$

Introducing this value of $\partial Z(p)/\partial p$ into (17), we get

$$\begin{aligned} \frac{\partial Z(p)}{\partial p} &= \frac{1}{2}nz_2RC \cosh n\gamma \\ &= \frac{nR}{2p} \cos (s\pi). \end{aligned} \tag{19}$$

For $p = 0$, find the value of $z_2 \sinh \gamma \sinh n\gamma$ as p approaches 0. We have

$$\cosh \gamma = 1 + \tfrac{1}{2}RCp;$$

as p approaches 0, $\cosh \gamma$ approaches unity, that is, γ is very small. Hence, we may write,

$$\cosh \gamma = 1 + \frac{\gamma^2}{2} = 1 + \frac{1}{2}RCp,$$

and

$$\gamma^2 = RCp.$$

Also, for very small values of γ,

$$\sinh \gamma = \gamma; \sinh n\gamma = n\gamma$$

We have, therefore, as p approaches zero,

$$z_2 \sinh \gamma \sinh n\gamma = \frac{n\gamma^2}{Cp} = \frac{nRCp}{Cp} = nR.$$

Hence,

$$Z(p)_{p=0} = nR. \tag{20}$$

Introducing the values of p and $\partial Z(p)/\partial p$ from (16), (19), and (20) in the expansion formula, we obtain the complete developed solution for the current in the nth section,

$$i_n = \frac{E}{nR} + E\sum \frac{2\epsilon^{-\left(\frac{4 \sin^2}{RC}\frac{s\pi}{2n}\right)t}}{nR \cos (s\pi)}. \tag{21}$$

When n is very large, $s\pi/2n$ small, $\sin s\pi/2n = s\pi/2n$, and the above reduces to the following:

$$i_n = \frac{E}{nR} + \frac{2E}{nR}\sum \cos\,(s\pi)\,\epsilon^{-\frac{s^2\pi^2}{RCn^2}t}. \tag{22}$$

Comparing this with formula (27), Chap. IV, derived for the current in a uniform cable, which is

$$i = \frac{E}{Rl} + \frac{2E}{Rl}\sum \cos\frac{n\pi x}{l}\epsilon^{-\frac{n^2\pi^2}{RCl^2}t},$$

we note that, for $x = l$, that is, at the end of the cable, the expression for the current is of the same form as (22), above. If, in one case, we consider Rl and Cl as the total resistance and total capacity of the cable and, in the artificial cable, Rn and Cn as the total resistance and capacity, then the two formulas are identical. Formula (22), however, was arrived at on the assumption that the angle $s\pi/2n$ is sufficiently small to permit replacing the sine of the angle by the angle itself; and this is what conditions the equivalence of the artificial cable to a uniform cable. The number of sections must be sufficiently large so that the sine of the angle $s\pi/2n$ has practically the same numerical value as the angle in circular measure.

NON-DISSIPATING ARTIFICIAL LINE

For an artificial cable of negligible resistance and leakage, the series and shunt elements are

$$z_1 = Lp;\ z_2 = \frac{1}{Cp}. \tag{23}$$

If n is large, the expression for the input current is as before,

$$i_0 = \frac{E}{z_2 \sinh\gamma}.$$

Also,

$$z_2 \sinh\gamma = \sqrt{z_1 z_2 + \tfrac{1}{4}z_1^2}.$$

Introducing the values of z_1 and z_2 given by (23), we have, for this case,

$$z_2 \sinh\gamma = \sqrt{\frac{L}{C} + \frac{1}{4}L^2p^2} = \frac{Lp}{2}\sqrt{1 + \frac{4}{LCp^2}}. \tag{24}$$

Hence,

$$\dot{i}_0 = \frac{2E}{Lp\sqrt{1 + \frac{4v^2}{p^2}}}, \tag{25}$$

$$v^2 = \frac{1}{LC}.$$

Expanding the denominator of (25), we obtain

$$\dot{i}_0 = \frac{2E}{Lp}\left\{1 - \frac{1}{2}\left(\frac{4v^2}{p^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{4v^2}{p^2}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{4v^2}{p^2}\right)^3 + \cdots\right\}$$

Bearing in mind the operational relation $1/p^n = t^n/n!$, the above transforms to

$$\dot{i}_0 = \frac{2E}{Lp}\left\{1 - \frac{1}{2}\frac{(4v^2)t^2}{2!} + \frac{1\cdot 3}{2\cdot 4}\frac{(4v^2)^2t^4}{4!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{(4v^2)^3t^6}{6!} + \cdots\right\}$$

which may be put in this form:

$$\dot{i}_0 = \frac{2E}{Lp}\left\{1 - \frac{(2vt)^2}{2^2} + \frac{(2vt)^4}{2^2\cdot 4^2} - \frac{(2vt)^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right\} \tag{26}$$

The bracket term is recognized as the series for the zero Bessel function and may, therefore, be put in the following convenient form:

$$\begin{aligned} \dot{i}_0 &= \frac{2E}{Lp} J_0(2vt) \\ &= \frac{2E}{L}\int_0^t J_0(2vt)dt. \end{aligned} \tag{27}$$

The expression for the current is given in the form of a definite integral of the Bessel function of the zero order.

To develop the solution for the current in the nth section, it is best to apply the expansion formula. The expression for the current in the nth section is

$$\dot{i}_n = \frac{E}{z_2 \sinh \sinh n\gamma}.$$

The determinantal equation

$$Z(p) = z_2 \sinh \gamma \sinh n\gamma = 0,$$

gives

$$\sinh n\gamma = 0,$$

and

$$n\gamma = js\pi;\ \gamma = j\frac{s\pi}{n}. \tag{28}$$

The values of p corresponding to these values of γ are obtained from the equation

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2}$$

For this case,

$$\cosh \gamma = \cos\left(\frac{s\pi}{n}\right) = 1 + \frac{1}{2}LCp^2. \tag{29}$$

Hence,

$$p = \pm j\sqrt{\frac{2}{LC}\left(1 - \cos\frac{s\pi}{n}\right)}$$

$$\pm j2v \sin\frac{s\pi}{2n}. \tag{30}$$

The expression for $\partial Z(p)/\partial p$ is readily obtained,

$$\frac{\partial Z(p)}{\partial p} = nz_2 \sinh \gamma \cosh n\gamma\frac{\partial\gamma}{\partial p}.$$

By (29)

$$\sinh \gamma\frac{\partial\gamma}{\partial p} = LCp.$$

Hence,

$$\frac{\partial Z(p)}{\partial p} = nz_2LC \cosh n\gamma$$

$$= nL \cos (s\pi). \tag{31}$$

To determine the value of $Z(p)$ for $p = 0$, take its value as p approaches zero. In this case,

$$\cosh \gamma = 1 + \frac{\gamma^2}{2} = 1 + \frac{1}{2}LCp^2,$$

and

$$\gamma^2 = LCp^2;$$

also,

$$\sinh \gamma \sinh n\gamma = n\gamma^2 = nLCp^2.$$

Hence,

$$Z(p)_{p=0} = z_2nLCp^2 = nLp = 0 \text{ for } p = 0.$$

This would indicate an infinite value of the steady-current component for applied steady voltage. This is a consequence of the fact that the resistances are entirely neglected. In actual practice, of course, the inevitable resistance of the coils would limit the magnitude of the steady-current component. We need not concern ourselves with it here. What we are interested

in mainly is to obtain the solution for the transient-current component.

Introducing the values of p and $\partial Z(p)/\partial p$ from (30) and (31) in the expansion formula, we get

$$i_n = E\sum\frac{\epsilon^{\pm j\left(2v\sin\frac{s\pi}{2n}\right)t}}{\pm j\,2v\sin\dfrac{s\pi}{2n}\cdot nL\cos(s\pi)}. \tag{32}$$

Taking both terms under the double sign in the above equation to include all the roots of the determinantal equation, it reduces to the following:

$$i_n = E\sum\frac{\sin\left\{vt\sin\dfrac{s\pi}{2n}\right\}}{nLv\cos(s\pi)\sin\dfrac{s\pi}{2n}}. \tag{33}$$

Equation (33) gives the complete solution to the problem of the current in the nth section, but it is in the form of a series which is not very convenient for numerical calculations. It is possible, however, to transform the solution to another form, that of a definite integral of a Bessel function, which is more suitable for computations.

We note that in the expansion formula each term of the summation may be replaced by a time definite integral as follows:

$$\int_0^t\frac{\epsilon^{p_st}}{\dfrac{\partial Z(p)}{\partial p}p = p_s}dt = \frac{\epsilon^{p_st}}{p_s\dfrac{\partial Z(p)}{\partial p}p = p_s} - \frac{1}{p_s\dfrac{\partial Z(p)}{\partial p}p = p_s},$$

hence,

$$\frac{\epsilon^{p_st}}{p_s\dfrac{\partial Z(p)}{\partial p}p = p_s} = \int_0^t\frac{\epsilon^{p_st}}{\dfrac{\partial Z(p)}{\partial p}p = p_s}dt + \frac{1}{p_s\dfrac{\partial Z(p)}{\partial p}p = p_s}. \tag{34}$$

Introducing the values of p_s and $\partial Z(p)/\partial p$ from (30) and (31) into the second right-hand term of (34), we get

$$\frac{1}{p_s\dfrac{\partial Z(p)}{\partial p}p = p_s} = \frac{1}{\pm nL\cos(s\pi)2jvt\sin\dfrac{s\pi}{2n}},$$

which equals to zero, taking into account the double sign.

Hence,

$$\frac{\epsilon^{p_s t}}{p_s \dfrac{\partial Z(p)}{\partial p}_{p = p_s}} = \int_0^t \frac{\epsilon^{p_s t}}{\dfrac{\partial Z(p)}{\partial p}_{p = p_s}} dt,$$

and using this in the expansion formula, we get

$$i_n = E \int_0^t \sum \frac{\epsilon^{p_s t}}{\dfrac{\partial Z(p)}{\partial p}_{p = p_s}} dt. \tag{35}$$

Introducing the values of p_s and $\partial Z(p)/\partial p$ from (30) and (31) we get

$$\begin{aligned} i_n &= E \int_0^t \sum \frac{\epsilon^{\pm j2vt} \sin \dfrac{s\pi}{2n}}{nL \cos (s\pi)} dt, \\ &= \frac{2E}{nL} \int_0^t \sum \frac{\cos \left(2vt \sin \dfrac{s\pi}{2n}\right)}{\cos (s\pi)} dt, \end{aligned} \tag{36}$$

the summation extending to all values of s from o to n, going by steps of π/n. If n is large, the summation term may be converted into a definite integral, thus:

Put

$$\frac{s\pi}{2n} = \theta; \ \frac{\pi}{2n} = d\theta; \ s\pi = 2n\theta,$$

and the summation term becomes a definite integral whose limits of integration are 0 and $\pi/2$, also multiplying numerator and denominator by cos $(s\pi)$ and remembering that $\cos^2 (s\pi) = 1$ for all values of s, equation (36) transforms into the following definite integral:

$$i_n = \frac{4E}{\pi L} \int_0^t \int_0^{\frac{\pi}{2}} \cos (2n\theta) \cos (2vt \sin \theta) d\theta dt. \tag{37}$$

The function cos $(2vt \sin \theta)$, however, can be expanded into a Furier-Bessel series,[1] thus:

$$\cos (2vt \sin \theta) = J_0(2vt) - 2J_2(2vt) \cos 2\theta + 2J_4(2vt) \cos 4\theta + \cdots \tag{38}$$

If we multiply both sides of (38) by cos $2n\theta$ and integrate from 0 to $\pi/2$, each term on the right-hand side except the term

[1] See GRAY and MATHEWS, "Treatise on Bessel Functions," p. 18.

$J_{2n}(2vt)\cos 2n\theta$ will be separately zero. Multiplying the term $J_{2n}(2vt)\cos 2n\theta$ by $\cos 2n\theta$, and integrating from 0 to $\pi/2$, gives

$$J_{2n}(2vt)\int_0^{\frac{\pi}{2}}\cos^2 2n\theta d\theta = \frac{J_{2n}^2(2vt)}{2n}\left[\frac{2n\theta}{2}+\frac{1}{4}\sin^4 n\theta\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{4}J_{2n}(2vt). \qquad (39)$$

We arrive, therefore, at the conclusion that

$$\int_0^{\frac{\pi}{2}}\cos 2n\theta\cos(2vt\sin\theta)d\theta = \frac{\pi}{4}J_{2n}(2vt).$$

Hence, substituting in (37) gives

$$i_n = \frac{E}{L}\int_0^t J_{2n}(2vt)dt. \qquad (40)$$

This is the equivalent of (33), expressed in an entirely different form, the integral of the Bessel function of the $2n$ order. The solution for the current in the zero section (27), arrived at by direct operational process, is of a similar form, the integral of the Bessel function of the zero order.

These formulas are more convenient for numerical calculations. There are extensive tables of the Bessel functions obtainable, from which the values of either $J_0(2vt)$ or $J_{2n}(2vt)$ for different values of t are readily obtained. Hence, in using these formulas, curves can be plotted, the Bessel functions as the ordinates and t as the abscissas extending from 0 to any desired value of t, and the area enclosed by these curves giving the integrals of these functions.

DISTORTIONLESS ARTIFICIAL LINE

A brief discussion of the properties and characteristics of a distortionless line is given in Chap. V. It is shown that if the electrical constants of the line are so related as to satisfy the condition $R/L = g/C$, then the line is said to be distortionless, that is, one on which electromagnetic waves are propagated without deformation. We shall consider here the problem of an artificial line in which the distortionless condition obtains, that is, one in which each series element consists of an inductance and

resistance, and each shunt element, of a capacity and leakage in parallel, and so related as to satisfy the condition

$$\frac{R}{L} = \frac{g}{C}. \tag{41}$$

If we put, for brevity,

$$\frac{R}{L} = \frac{g}{C} = 2a, \tag{42}$$

then we have

$$\left.\begin{aligned} z_1 &= Lp + R = L(p + 2a), \\ z_2 &= \frac{1}{Cp + g} = \frac{1}{C(p + 2a)}. \end{aligned}\right\} \tag{43}$$

It may be observed in passing, that if the artificial cable closely simulates a uniform cable, which is realized when the number of

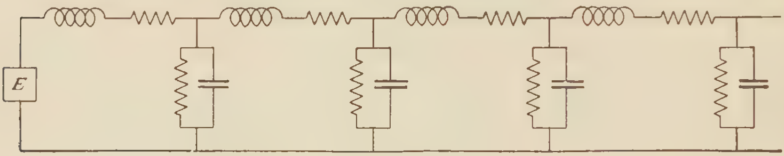

FIG. 24.

sections is large, then the propagation constant γ should be the same as the propagation constant of a uniform line, which we designated by K in Chap. V,

$$K = \sqrt{(Lp + R)(Cp + g)}.$$

When the number of sections is large, L, C, R, g per section are very small, and, in that case, from the relation

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2} = 1 + \frac{1}{2}(Lp + R)(Cp + g),$$

it is evident that $\cosh \gamma$ differs from unity by a small quantity; that is, γ is very small. We may, therefore, put

$$\cosh \gamma = 1 + \tfrac{1}{2}\gamma^2 = 1 + \tfrac{1}{2}(Lp + R)(g + Cp),$$

and

$$\gamma = \sqrt{(Lp + R)(g + Cp)},$$

the same as K.

We shall confine the discussion, in this case, to the determination of the current in the nth section. We have here, as in the previous case, the general expression

$$i_n = \frac{E}{z_2 \sinh \gamma \sinh n\gamma}.$$

The determinantal equation is, in this case, also,

$$\sinh n\gamma = 0,$$

and

$$n\gamma = js\pi;\ \gamma = j\frac{s\pi}{n}.$$

The values of p corresponding to the roots of the determinantal equation are determined from the relation

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2},$$

which, for this case, gives

$$\cosh \gamma = \cos \frac{s\pi}{n} = 1 + \frac{1}{2}LC(p + 2a)^2. \tag{44}$$

Hence,

$$p = -2a \pm jv\sqrt{2\left(1 - \cos \frac{s\pi}{n}\right)},$$

$$= -2a \pm j2v \sin \frac{s\pi}{2n}. \tag{45}$$

We also have here

$$\frac{\partial Z(p)}{\partial p} = nz_2 \sinh \gamma \cosh n\gamma \frac{\partial \gamma}{\partial p}. \tag{46}$$

By (44),

$$\sinh \gamma \frac{\partial \gamma}{\partial p} = LC(p + 2a)$$

Introducing this value in (46), we get

$$\frac{\partial Z(p)}{\partial p} = nz_2LC(p + 2a) \cosh n\gamma$$

$$= nL \cos (s\pi). \tag{47}$$

For $p = 0$, $Z(p)_{p=0} = z_2' \sinh \gamma' \sinh n\gamma'$ where $z_2' = 1/2Ca$; and γ' is determined from the relation

$$\cosh \gamma' = 1 + 2LCa^2.$$

In the expansion formula, $E/Z(p)_{p=0}$ gives the permanent-current component. This, however, is not of any particular interest here. What concerns us chiefly is the transient-current component. This is given in accordance with the expansion theorem by

$$i_n = E\sum_p \frac{\epsilon^{pt}}{p\frac{\partial Z(p)}{\partial p}}.$$

Substituting the values of p and $\partial Z(p)/\partial p$ from (45) and (47) results in the following:

$$i_n = E\epsilon^{-2at}\sum \frac{\epsilon^{\pm j\left(2vt\sin\frac{s\pi}{2n}\right)}}{nL\cos(s\pi)\left\{\pm j2v\sin\frac{s\pi}{2n} - 2a\right\}}. \tag{48}$$

Taking both terms under the double sign to include all the roots of the determinant, the above is expanded into

$$i_n = \frac{E\epsilon^{-2at}}{nL}\left\{\sum \frac{\cos\left(2vt\sin\frac{s\pi}{2n}\right) + j\sin\left(2vt\sin\frac{s\pi}{2n}\right)}{\left(j2v\sin\frac{s\pi}{2n} - 2a\right)\cos(s\pi)} - \sum \frac{\cos\left(2vt\sin\frac{s\pi}{2n}\right) - j\sin\left(2vt\sin\frac{s\pi}{2n}\right)}{\left(j2v\sin\frac{s\pi}{2n} + 2a\right)\cos(s\pi)},\right\} \tag{49}$$

and this simplifies to

$$i_n = \frac{2E\epsilon^{-2at}}{nL}\sum \frac{\left(2v\sin\frac{s\pi}{2n}\right)\sin\left(2vt\sin\frac{s\pi}{2n}\right) - 2a\cos\left(2vt\sin\frac{s\pi}{2n}\right)}{\left(4v^2\sin^2\frac{s\pi}{2n} + 4a^2\right)\cos(s\pi)}. \tag{50}$$

This is the complete developed solution for the transient current in the nth section. The solution is in the form of a series which is rather complicated and would be very laborious to use in computations. It is possible, however, in this case, also, to obtain a solution in the form of a definite integral of a Bessel function, as in the case of the problem of the non-dissipative artificial line. As in the previous case, we can replace $\dfrac{\epsilon^{p_s t}}{p_s \dfrac{\partial Z(p)}{\partial p}}$ by $\displaystyle\int_0^t \frac{\epsilon^{p\,t}}{\dfrac{\partial Z(p)}{\partial p}}dt$ and write

$$i_n = E\int_0^t \sum \frac{\epsilon^{pt}}{\dfrac{\partial Z(p)}{\partial p}}dt. \tag{51}$$

Introducing the values of p and $\partial Z(p)/\partial p$, we get

$$i_n = E\int_0^t \sum \frac{\epsilon^{-at}\epsilon^{\pm j\left(2vt\sin\frac{s\pi}{2n}\right)}}{nL\cos(s\pi)}dt,$$

$$= \frac{2E}{nL}\int_0^t \sum \frac{\epsilon^{-at}\cos\left(2vt\sin\frac{s\pi}{2n}\right)}{\cos(s\pi)}dt. \tag{52}$$

We have, however, already established that

$$\sum \frac{2\cos\left(2vt\sin\frac{s\pi}{2n}\right)}{n\cos(s\pi)} = J_{2n}(2vt)$$

(see equations (36) to (40)), therefore,

$$i_n = \frac{E}{L}\int_0^t \epsilon^{-at}J_{2n}(2vt)dt, \tag{53}$$

the same as (40) modified by an attenuation factor ϵ^{-at}.

Series Elements: Inductance and Resistance. Shunt Elements: Capacities.—The investigation of the properties and characteristics of an artificial line in which the leakage elements are not included is of considerable importance, inasmuch as this is the condition resembling more closely an ordinary transmission line in which leakage is generally a negligible factor. Also, this type of artificial line is very closely related in its properties and performance to the periodically loaded line. The solution of this problem, therefore, will serve also as an introduction to the study of the loaded-line problem. For a comprehensive mathematical discussion of the loaded line, reference must be had to Dr. Pupin's celebrated papers on this subject.[1]

For an artificial line of negligible leakage

$$\left.\begin{aligned} z_1 &= Lp + R, \\ z_2 &= \frac{1}{Cp}, \end{aligned}\right\} \tag{54}$$

and

$$\cosh\gamma = 1 + \frac{1}{2}\frac{z_1}{z_2} = 1 + \frac{1}{2}Cp(Lp + R). \tag{55}$$

[1] *Trans. Am. Inst. Elec. Eng.*, Vol. XVI, 1899; Vol. XVII, 1900.

The determinantal equation is, of course, the same as in the previous cases. Hence, we have, as before,

$$\gamma = j\frac{s\pi}{n}.$$

We have, therefore,

$$\cos\frac{s\pi}{n} = 1 + \frac{1}{2}(LCp^2 + RCp),$$

from which the values of p are determined as follows:

$$p_s = -\frac{R}{2L} \pm \frac{\sqrt{R^2C^2 - 8LC\left(1 - \cos\frac{s\pi}{n}\right)}}{2LC}$$

$$= -a \pm j\sqrt{4v^2\sin^2\frac{s\pi}{2n} - a^2}, \qquad (56)$$

where, for brevity,

$$a = \frac{R}{2L};\ v = \frac{1}{\sqrt{LC}}.$$

We also have, as before,

$$\frac{\partial Z(p)}{\partial p} = nz_2 \sinh\gamma \cosh n\gamma\frac{\partial\gamma}{\partial p},$$

but, in this case,

$$\sinh\gamma\frac{\partial\gamma}{\partial p} = LCp + \frac{1}{2}RC.$$

Hence,

$$\frac{\partial Z(p)}{\partial p} = nz_2\left(LCp + \frac{1}{2}RC\right)\cosh n\gamma$$

$$= \frac{nL}{p}(p + a)\cos(s\pi). \qquad (57)$$

For $p = 0$, we take the value of $Z(p) = z_2 \sinh\gamma \sinh n\gamma$ as p approaches zero. For small values of p, $\cosh\gamma$ differs from unity by a very small quantity only, and we may, therefore, put

$$\cosh\gamma = 1 + \tfrac{1}{2}\gamma^2 = 1 + \tfrac{1}{2}(LCp^2 + RCp)$$

Therefore,

$$\gamma^2 = LCp^2 + RCp,$$

and

$$Z(p)_{p=0} = z_2\sinh\gamma\sinh n\gamma = \frac{1}{Cp}n\gamma^2 = \frac{n(LCp^2 + RCp)}{Cp}$$

$$nR. \qquad (58)$$

That is, the steady-state component is simply E/nR. For the transient-current component, we have

$$i_n = E\sum \frac{\epsilon^{pt}}{p\dfrac{\partial Z(p)}{\partial p}}$$

$$= E\sum \frac{\epsilon^{pt}}{nL(p+a)\cos(s\pi)}. \tag{59}$$

Introducing the values of p from (56), we get

$$i_n = E\epsilon^{-at}\sum \frac{\epsilon^{\pm j\sqrt{4v^2\sin^2\frac{s\pi}{2n} - a^2}\,t}}{\pm jnL\cos(s\pi)\sqrt{4v^2\sin^2\dfrac{s\pi}{2n} - a^2}}. \tag{60}$$

Taking both terms under the double signs to include all the values of p corresponding to the roots of the determinantal equation, the above simplifies to the following:

$$i_n = \frac{2E\epsilon^{-at}}{nL}\sum \frac{\sin\left(\sqrt{4v^2\sin^2\dfrac{s\pi}{2n} - a^2}\right)t}{\cos(s\pi)\sqrt{4v^2\sin^2\dfrac{s\pi}{2n} - a^2}}. \tag{61}$$

The current is the sum of n oscillatory components, all having the same damping factor, but each of different frequency. The frequencies of oscillations are given by

$$f_s = \frac{1}{2\pi}\sqrt{4v^2\sin^2\frac{s\pi}{2n} - a^2}. \tag{62}$$

If n is infinitely large, the condition of a uniform line is approached, and in that case the solution given by (61) should reduce to that of the uniform line. That this is actually the case can be readily verified by comparing the above solution with (80), Chap. V, the solution for the oscillations on a uniform line.

If n is very large, we may put

$$\sin\frac{s\pi}{2n} = \frac{s\pi}{2n},$$

and (61) reduces to

$$i_n = \frac{2E\epsilon^{-at}}{nL}\sum \frac{\sin\left(\sqrt{\dfrac{v^2s^2\pi^2}{n^2} - a^2}\right)t}{\cos(s\pi)\sqrt{\dfrac{v^2s^2\pi^2}{n^2} - a^2}}. \tag{63}$$

Referring back to equation (80), Chap. V, we note that, neglecting leakage and putting $x = l$, the expression for the current at the end of the line reduces to the following:

$$i_l = \frac{2Ev^2}{l} C\epsilon^{-\rho t} \sum \frac{\sin \beta_n t}{\beta_n} \cos (n\pi) \cdot \tag{64}$$

For $g = 0$;

$$\rho = \frac{R}{2L} = a,$$

$$\beta_n = \sqrt{\frac{n^2\pi^2v^2}{l^2} - a^2} \cdot$$

$$\sigma = \frac{R}{2L} = a,$$

and

$$i_l = \frac{2E}{Ll} \epsilon^{-at} \sum \frac{\sin \left(\sqrt{\frac{n^2\pi^2v^2}{l^2} - a^2} \right) t}{\sqrt{\frac{n^2\pi^2v^2}{l^2} - a^2}} \cos (n\pi) \cdot \tag{65}$$

If we bear in mind that n, the total number of sections in (61), is the equivalent of l, the length of the line, in (63), and that s in (61) and n in (63) designate the same thing—the summation steps—then it is evident that the two formulas are identical. The solution for the uniform line can be derived from that of the artificial line by making the number of sections infinitely large. The solution of the uniform line is a limiting case of that of the artificial line.

ARTIFICIAL LINE-SERIES ELEMENTS INDUCTANCE AND RESISTANCE; SHUNT ELEMENTS CAPACITY AND LEAKAGE

The basic formula (13) for the current in the nth section applies, also, of course, to this case; that is,

$$i_n = \frac{E}{z_2 \sinh \gamma \sinh n\gamma}, \tag{13 bis}$$

and

$$\gamma = j\frac{s\pi}{n} \cdot$$

The values of p to be used in the expansion formula are here also determined from the relation

$$\cosh \gamma = 1 + \frac{1}{2}\frac{z_1}{z_2} \cdot$$

In this case, however, $z_1 = (Lp + R)$ and $z_2 = \dfrac{1}{Cp + g}$, hence,

$$\cosh \gamma = \cos\left(\frac{s\pi}{n}\right) = 1 + \frac{1}{2}(Lp + R)(Cp + g), \tag{66}$$

which may be put in this form:

$$\cos\left(\frac{s\pi}{n}\right) = 1 + \frac{1}{2v^2}\{p^2 + 2(a + b)p + 4ab\}\cdot$$

Solving for p, we obtain

$$\begin{aligned} p &= \frac{-2(a + b) \pm \sqrt{4(a + b)^2 - 16ab - 8v^2\left(1 - \cos\frac{s\pi}{n}\right)}}{2} \\ &= -(a + b) \pm \sqrt{(a - b)^2 - 4v^2 \sin^2 \frac{s\pi}{2n}}, \\ &= -\rho \pm j\sqrt{4v^2 \sin^2 \frac{s\pi}{2n} - \sigma^2}\cdot \end{aligned} \tag{67}$$

We may put, for brevity,

$$p = -\rho \pm j\beta_s,$$

where

$$\beta_s = \sqrt{4v^2 \sin \frac{s\pi}{2n} - \sigma^2}\cdot \tag{68}$$

The expression for $\partial Z(p)/\partial p$ to be used in the expansion formula is readily obtained,

$$\frac{\partial Z(p)}{\partial p} = z_2 \sinh \gamma \cosh n\gamma \frac{\partial(n\gamma)}{\partial p}\cdot \tag{69}$$

By (66), however,

$$\begin{aligned} \text{Sinh } \gamma\frac{\partial \gamma}{\partial p} &= \frac{1}{2}(2LCp + Lg + RC) \\ &= \frac{1}{v^2}(p + a + b) = \frac{1}{v^2}(p + \rho)\cdot \end{aligned} \tag{70}$$

Introducing this value in (69), we get

$$\begin{aligned} \frac{\partial Z(p)}{\partial p} &= nz_2 \frac{1}{v^2} (p + \rho) \cosh n\gamma \\ &= \frac{n(p + \rho)}{v^2(Cp + g)} \cos (sn) \\ &= \frac{\pm nj\beta_s}{Cv^2(-\rho + j\beta_s + 2b)} \cos (s\pi) = \frac{\pm nj\beta_s \cos (s\pi)}{v^2C(-\sigma \pm j\beta_s)}\cdot \end{aligned} \tag{71}$$

Substituting these values of p and $\partial Z(p)/\partial p$ in the expansion formula, we obtain the following:

$$i_n = \frac{E}{nL}\sum \frac{(-\sigma \pm j\beta_s)\epsilon^{(-\rho \pm j\beta_s)t}}{\pm j\beta_s(-\rho \pm j\beta_s)\cos(s\pi)}. \tag{72}$$

Taking into account the double-sign terms to include the values of p corresponding to all the roots of the determinantal equation, we obtain the complete expression, as follows:

$$i_n = \frac{E}{nL}\epsilon^{-\rho t}\sum \frac{(-\sigma + j\beta_s)\epsilon^{j\beta_s t}}{j\beta_s(-\rho + j\beta_s)\cos(s\pi)} - \frac{(\sigma + j\beta_s)\epsilon^{-j\beta_s t}}{j\beta_s(\rho + j\beta_s)\cos(s\pi)},$$

$$= \frac{E}{nL}\epsilon^{-\rho t}\sum \frac{(-\sigma + j\beta_s)(-\rho - j\beta_s)\epsilon^{j\beta_s t} - (\sigma + j\beta_s)(\rho - j\beta_s)\epsilon^{-j\beta_s t}}{j\beta_s(\rho^2 + \beta_s^2)\cos(s\pi)}$$

$$= \frac{2E\epsilon^{-\rho t}}{nL}\sum \frac{\beta_s(\sigma - \rho)\cos\beta_s t + (\sigma\rho + \beta_s^2)\sin\beta_s t}{\beta_s(\rho^2 + \beta_s^2)\cos(s\pi)}. \tag{73}$$

This is the complete solution for the transient-current component in the last, the nth, section of an artificial line which is to simulate a line in which all the electrical constants, R, L, C, g, are active.

Equation (73) may be put in a little more convenient form, thus:

$$i_n = \frac{2E\epsilon^{-\rho t}}{nL}\sum \sqrt{\frac{\sigma^2 + \beta_s^2}{\rho^2 + \beta_s^2}}\frac{\cos(\beta_s t - \varphi_s)}{\beta_s\cos(s\pi)}. \tag{74}$$

$$\tan\varphi_s = \frac{\sigma\rho + \beta_s^2}{(\sigma - \rho)\beta_s}. \tag{75}$$

Formula (73) covers the most general case and should reduce to the special cases discussed before by the proper choice of values of R, C, L, g.

For $R = g = 0$, the formula should reduce to (33), the expression for the current in a non-dissipative line. For this condition, $\rho = 0$, $\sigma = 0$; $\beta = 2v\sin\frac{s\pi}{2n}$ and (73) reduces to

$$i_n = \frac{2E}{nL}\sum \frac{\sin\left(2vt\sin\frac{s\pi}{2n}\right)}{2v\sin\frac{s\pi}{2n}\cos(s\pi)},$$

which is formula (33).

For the distortionless-line condition, $\sigma = 0$, $\rho = 2a$, $\beta = 2v\sin\frac{s\pi}{2n}$; and for these values (73) reduces to (50).

For $g = 0$, $\rho = 2a$; $\sigma = 2a$ and (73) reduces to (61).

CHAPTER VII

HEAVISIDE'S DERIVATION OF EXPANSION FORMULA

It will be of interest to give an outline here of the method by which Heaviside arrived at the expansion formula. Aside from the historical interest, the physical and mathematical ideas by which he was led to the formulation of this important theorem should be of considerable interest for the student of physics and engineering. Heaviside was profoundly interested in the study of the dynamics of an electromagnetic field; there are frequent recurrences to this subject all through his collected papers. Irrespective of what particular investigation in electrical theory he was working on, the subject of the dynamics of an electromagnetic field, and related matters, is touched upon from one angle or another. It was the study of energy distribution and subsidence in an electromagnetic system that suggested ideas which led to the formulation of the expansion theorem. Nowhere has he given a complete and formal demonstration of the derivation of the formula. But ideas and suggestions relating to this and connected problems run through the publications of his collected papers.

In his "Electromagnetic Theory,"[1] he gives the formula without proof. He contents himself with the following statement, and no reference is made to the process of derivation:

> Finally, there is a third and very general way of converting operational solutions to the form of the sum of normal solutions. It does not require special investigations of the properties of normal functions. It is very direct and uniform of application. It avoids, in general, a large amount of unnecessary work. The investigation of the conjugate property, and of the terminal apparatus in detail in order to apply it to the determination of the coefficients, is wholly avoided. It applied to all kinds of series of normal functions, as well as Fourier series. And it applies generally in electromagnetic problems, with a finite or infinite number of variables; or more generally, to the system of dynamical equations used by Lord Rayleigh in the first volume of his treatise on

[1] Vol. II, p. 127

Sound, which covers the rest of the work, and upon which he bases his discussion of general properties.

The method may be briefly (though imperfectly) stated as follows: Let $e = ZC$ be the operational solutions of an electromagnetic problem; say, for definiteness, that C is the current at a certain place due to an impressed force e at the same or some other place. Let the form of Z be such as to indicate the existence of normal solutions for C. Then when e is steady, beginning at the moment $t = 0$, the C due to e is expressed by

$$C = \frac{l}{Z_0} + e\sum_p \frac{\epsilon^{pt}}{p\frac{\partial Z}{\partial p}}.$$

This is recognized as the expansion formula which we have developed by algebraic process; the complete derivation is given in Chapter II.

Because of the fact that in giving this formula in his electromagnetic theory, no reference is made to his collected papers, nor is any hint given as to the how and when of its derivation, the impression was created that Heaviside gave this formula without proof. This, of course, is incorrect. Heaviside discussed the subject very extensively but in scattered form.[1]

The basic ideas which led to the development of the expansion formula relate to questions of energy distribution and subsidence in a normal electromagnetic system. A normal system is defined as one which on subsiding remains similar to itself, the subsidence being represented by the time factor ϵ^{pt}.

If we have any electric-circuit system, energy stored up magnetically and electrically in the inductances and condensers, and leave it to itself, removing the impressed force, it will subside to equilibrium in a manner determined by the distribution of the currents in the coils and the charges of the condensers. In general, the energy relations are as follows:

$$W + \frac{d}{dt}(U + T) = 0, \tag{1}$$

where U is the electric energy T the magnetic energy, and W the energy dissipated. This formula states that the rate of decrease

[1] It is partially dealt with in his papers, "Electromagnetic Induction and Its Propagation," collected papers, Vol. I, pp. 429–560; "On Self Induction of Wires," collected papers, Vol. II, pp. 168–323; and the final derivation of the formula is given in his paper "Resistance and Conductance Operators," collected papers, Vol. II, pp. 355–374.

of the electric and magnetic energies of the system is equal to the dissipativity.

When the energy subsidence occurs in two normal systems, as in the case, for instance, of a double periodic oscillating circuit system, and if we designate the two systems by suffixes $_1$ and $_2$, an additional energy relation is given for this case by the equation

$$W_{12} + \frac{d}{dt}(U_{12} + T_{12}) = 0, \qquad (2)$$

an equation of mutual acticity. U_{12} and T_{12} are the mutual electric and magnetic energies, and W_{12} the mutual dissipativity. The mutual energies are the excess of the total energies when the two normal systems coexist over the sum of the separate energies if they would have existed independently. In addition, for a system in which no energy is communicated, leaving it only irreversibly through dissipativity, there is also the relation

$$U_{12} = T_{12}. \qquad (3)$$

The mutual electric and magnetic energies are equal. Heaviside established these relations from general considerations of a dynamical system, which will be given farther on. We shall first consider these questions from the electric-circuit standpoint, following the method of Wagner.[1]

Take the simplest case, that of two circuite coupled through a common element, and consider the three possible cases—capacity coupling, inductance coupling, and resistance coupling.

a. Capacity Coupling.—The circuit arrangement is shown in Fig. 25; the circuits are coupled through the condenser C_{12}.

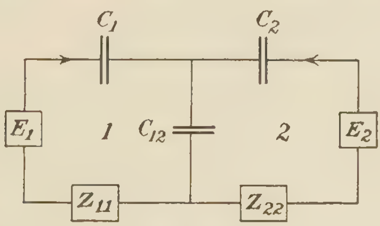

Fig. 25.

If the charge on condenser C_1 is Q_1, and the charge on condenser C_2 is Q_2, the charge on the coupling condenser C_{12} is $Q_1 + Q_2$.

[1] Wagner, Karl Willy, "Der Satz von der Wechselseitingen Energie," *Elektrische Nachrichten-Technik*, Bd. 2, S. 376–392.

Z_{11} and Z_{22} are the impedances of the circuits 1 and 2 exclusive of capacities. We have the circuit equations

$$\left.\begin{aligned} \frac{Q_1}{C_1} + I_1Z_{11} + \frac{Q_1 + Q_2}{C_{12}} &= E_1, \\ \frac{Q_2}{C_2} + I_2Z_{22} + \frac{Q_1 + Q_2}{C_{12}} &= E_2. \end{aligned}\right\} \tag{4}$$

Put

$$\left.\begin{aligned} K_{11} &= \frac{1}{C_1} + \frac{1}{C_{12}}, \\ K_{22} &= \frac{1}{C_2} + \frac{1}{C_{12}}, \\ K_{12} &= K_{21} = \frac{1}{C_{12}}. \end{aligned}\right\} \tag{5}$$

Also,

$$I_1 = \frac{dQ_1}{dt} = pQ_1;\ I_2 = \frac{dQ_2}{dt} = pQ_2.$$

Substituting these in equations (4), we get the following:

$$\left.\begin{aligned} (K_{11} + pZ_{11})Q_1 + K_{12}Q_2 &= E_1, \\ (K_{22} + pZ_{22})Q_2 + K_{21}Q_1 &= E_2. \end{aligned}\right\} \tag{6}$$

b. *Inductive Coupling.*—Two circuite coupled through an inductance L_{12}, the arrangement shown in Fig. 26. If we put

$$L_{11} = L_1 + L_{12}$$

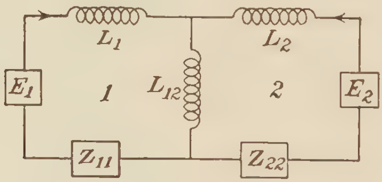

Fig. 26.

the total inductance of circuit 1, and

$$L_{22} = L_2 + L_{12}$$

the total inductance of circuit 2, we have the following equations for the two circuits:

$$\left.\begin{aligned} (pZ_{11} + p^2L_{11})Q_1 + p^2L_{12}Q_2 &= E_1, \\ (pZ_{22} + p^2L_{22})Q_2 + p^2L_{21}Q_1 &= E_2, \end{aligned}\right\} \tag{7}$$

Z_{11} and Z_{22} are the impedances of the circuits exclusive of inductances.

c. Resistance Coupling.—The circuit arrangement is shown in Fig. 27. R_{12} is the coupling resistance, Z_{11} and Z_{22} are the impedances of the circuits exclusive of resistances. If we put $R_{11} = R_1 + R_{12}$, the total resistance of circuit 1, and $R_{22} = R_2 + R_{21}$ the total resistance of circuit 2, the circuit equations will be as follows:

$$\left.\begin{aligned} p(Z_{11} + R_{11})\mathrm{Q}_1 + pR_{12}\mathrm{Q}_2 &= E_1, \\ p(Z_{22} - E_{22})Q_2 + pR_{12}Q_1 &= E_2. \end{aligned}\right\} \tag{8}$$

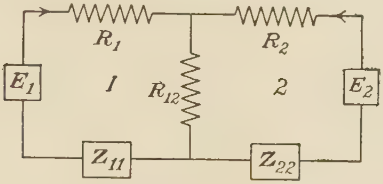

FIG. 27.

In the general case for any number of circuits and all three types of coupling simultaneously effective between the circuits, the circuit equations in accordance with the preceding will be as follows:

$$\left.\begin{aligned} (K_{11} + pR_{11} + p^2L_{11})Q_1 + (K_{12} + pR_{12} + p^2L_{12})Q_2 + \cdots \\ = E_1, \\ (K_{22} + pR_{22} + p^2L_{22})Q_2 + (K_{21} + pR_{21} + p^2L_{21})Q_1 + \cdots \\ = E_2. \end{aligned}\right\} \tag{9}$$

We also have the following expressions for the energies in the electric circuit system.

The electric energy in the condensers

$$U = \tfrac{1}{2}K_{11}Q_1^2 + K_{12}Q_1Q_2 + \tfrac{1}{2}K_{22}Q_2^2 + \cdots \tag{10}$$

The magnetic energy in the inductances

$$T = \tfrac{1}{2}L_{11}T_1^2 + L_{12}I_1I_2 + \tfrac{1}{2}L_{22}I_2^2 + \cdots \tag{11}$$

The energy dissipated by the resistances

$$W = R_{11}I_1^2 + 2R_{12}I_1I_2 + R_{22}I_2^2 + \cdots \tag{12}$$

The power, rate of energy, supplied to the circuits by the impressed electromotive forces,

$$S = E_1I_1 + E_2I_2 + \cdots \tag{13}$$

By the aid of the above formulæ, we can establish the relation between mutual electric and magnetic energies.

Assume two systems of electromotive forces acting on a circuit network. One system we shall distinguish by $'$ thus: E'_1, E'_2, . . . , the corresponding charges by Q'_1, Q'_2, . . . and the corresponding currents by I'_1, I'_2, . . . We shall also designate by U' the electric energy; T' the magnetic energy; W' the dissipativity; and S' the power supplied.

A second system of electromotive forces, different from the first, we shall distinguish by $''$ thus: E''_1, E''_2 . . . ; the charges corresponding to these electromotive forces are designated by Q''_1, Q''_2, . . . , the currents by I''_1, I''_2, . . . ; also designated by U'', the electric energy, T'' the magnetic energy, W'' the dissipativity, and S'' the power supplied.

When both systems of electromotive forces act simultaneously on the circuit network, the charges and the currents are additive so the effective charges are $Q'_1 + Q''_1$, $Q'_2 + Q''_2$ . . . and the effective currents are $I'_1 + I''_1$, $I_2 + I''_2$, . . . The energies, however, of the two systems are not additive, because the energies are proportional to the squares of the charges and the currents. The resultant energies are as follows:

$$\left.\begin{aligned} U &= U' + U'' + U_{12}, \\ T &= T' + T'' + T_{12}, \\ W &= W' + W'' + W_{12}, \\ S &= S' + S'' + S_{12}. \end{aligned}\right\} \tag{14}$$

U_{12} designates the mutual electric energy between systems 1 and 2,

$$U_{12} = K_{11}Q_1Q''_1 + K_{12}(Q'_1Q''_2 + Q''Q'_2) + K_{22}Q'_2Q''_2 + \cdots \tag{15}$$

T_{12} is the mutual magnetic energy between systems 1 and 2.

$$T_{12} = L_{11}I'_1I''_1 + L_{12}(I'_1I'' + I'_1I''_2) + L_{22}I'_2I''_2 + \cdots \tag{16}$$

The corresponding mutual dissipativity is

$$W_{12} = 2\{R_{11}T'_1T'' + R_{12}(I'_1I''_2 + I''_1I'_2) + R_{22}I'_2I''_2\} + \cdots \tag{17}$$

The corresponding power, rate of energy, supply is given by

$$S_{12} = (E'_1I''_1 + E''_1I'_1) + (E'_2I''_2 + E''_2I'_2) + . . . \tag{18}$$

Of the mutual power supply, the part contributed by system 1 is

$$S'_{12} = E'_1I''_1 + E'_2I''_2 + \cdots \tag{19}$$

We will now assume that in system 1, all amplitudes vary as the exponential function ϵ^{p_1t} and that in system 2, all amplitudes

vary as the exponential function $\epsilon^{p_2 t}$. We shall have then, by (9),

$$\left.\begin{array}{r} k_{11}Q_1' + R_{11}I_1' + p_1L_{11}I_1' + k_{12}Q_2' + R_{12}I_2' + p_1L_{12}I_2' + \\ \cdots = E_1' \end{array}\right\} \quad (20)$$

$$I_2'' = p_2Q_1'', \text{ etc.}$$

Introducing these values in (19), we obtain

$$S_{12}' = p_2U_{12} + p_1T_{12} + \tfrac{1}{2}W_{12}. \quad (21)$$

In a similar way, from the relation

$$S_{21}'' = E_1''I_1 + E_2''I_2' + \cdots$$

we get

$$S_{12}'' = p_1U_{12} + p_2T_{12} + \tfrac{1}{2}W_{12}. \quad (22)$$

These are the equations of mutual activity. Subtracting (22) from (21), we get

$$S_{12}' - S_{21}'' = (p_2 - p_1)(U_{12} - T_{12}). \quad (23)$$

Now let the E's vanish, so that no energy can be communicated to the system, then $S_{12}' = 0$, and $S_{12}'' = 0$, and

$$(p_2 = p_1)(U_{12} - T_{12}) = 0,$$

giving

$$U_{12} = T_{12}. \quad (24)$$

By the aid of this relation between the mutual energies, the amplitudes of the normal functions of a system of any number of degrees of freedom can be determined. The expansion formula was derived by the application of this principle to electric-circuit systems.

Heaviside established this relation between the mutual energies from the general considerations of a dynamical system. He states the whole problem very clearly and concisely in his paper[1] as follows:

THE CONJUGATE PROPERTY $U_{12} = T_{12}$ IN A DYNAMICAL SYSTEM WITH LINEAR CONNECTIONS

Considering only a dynamical system in which the forces of reaction are proportional to displacements, and the forces of resistance to velocities, there are three important quantities—the potential energy, the kinetic energy, and the dissipativity, say U, T, and Q which are quadratic functions of the variables or their velocities. When there is no kinetic energy, the conjugate properties of normal systems are $U_{12} = 0$ and $Q_{12} = 0$, these standing for the mutual potential energy

[1] "On the Self Induction of Wires," collected papers Vol. II, p. 202.

and the mutual dissipativity of a pair of normal systems. When there is no potential energy, we have $T_{12} = 0$ and $Q_{12} = 0$. When there is no dissipation of energy, $U_{12} = 0$ and $T_{12} = 0$. And, in general, $U_{12} = T_{12}$, which covers all cases and has two equivalents, $\frac{1}{2}Q_{12} + \dot{U}_{12} = 0$, and $\frac{1}{2}Q_{12} + \dot{T}_{12} = 0$; for, as the mutual potential and kinetic energies are equal, the mutual dissipativity is derived half from each.

Let the variables be $x_1, x_2 \cdots$, their velocities $v_1 = \dot{x}_1 \cdots$ and the equations of motion

$$\left.\begin{aligned} F_1 &= (A_{11} + B_{11}p + C_{11}p^2)\,x_1 + (A_{12} + B_{12}p + C_{12}p^2)\,x_2 + \cdots \\ F_2 &= (A_{21} + B_{21}p + C_{21}p^2)\,x_1 + (A_{22} + B_{22}p + C_{22}p^2)\,x_2 + \cdots \end{aligned}\right\} \quad (88)$$

where $F_1, F_2 \cdots$ are impressed forces and p stands for d/dt. Forming the equation of total activity, we obtain

$$\Sigma\, Fv = Q + \dot{U} + \dot{T} \qquad (89)$$

where

$$\left.\begin{aligned} 2U &= A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2 + \cdots, \\ Q &= B_{11}v_1^2 + 2B_{12}v_1v_2 + B_{22}v_2^2 + \cdots, \\ 2T &= C_{11}v_1^2 + 2C_{12}v_1v_2 + C_{22}v_2^2 + \cdots. \end{aligned}\right\} \quad (90)$$

So far will define in the briefest manner, U, T, Q and activity.

Now let the F's vanish, so that no energy can be communicated to the system, whilst it can only leave it irreversibly, through Q. Then let p_1, p_2 be any two values of p satisfying (88) regarded as algebraic. Let Q_1, U_1, T_1 belong to the system p_1 existing alone; then, by (89) and (90),

$$\begin{aligned} 0 &= Q_1 + \dot{U}_1 + \dot{T}_1, \text{ or } 0 = Q_1 + 2p_1(U_1 + T_1); \\ 0 &= Q_2 + \dot{U}_2 + \dot{T}_2, \text{ or } 0 = Q_2 + 2p_2(U_2 + T_2). \end{aligned}$$

But when existing simultaneously, so that

$$Q = Q_1 + Q_2 + Q_{12},\ U = U_1 + U_2 + U_{12},\ T = T_1 + T_2 + T_{12},$$

where U_{12}, T_{12}, Q_{12} depend upon products from both systems, thus:

$$\begin{aligned} Q_{12} &= 2\{B_{11}v_1v_1' + B_2v_2v_2' + B_{12}(v_1v_2' + v_2v)_1' + \cdots\} \\ U_{12} &= A_{11}x_1x_1' + A_2x_2x_2' + A_{12}(x_1x_2' + x_2x_1') + \cdots \\ T_{12} &= C_{11}v_1v_1' + C_{22}v_2^2v_2^2 + C_{12}(v_1v_2' + v_2v_1') + \cdots, \end{aligned}$$

the accents distinguishing one system from the other, we shall find, by forming the equations of mutual activity, $\Sigma F_v' = \cdots$, and $\Sigma F_v' = \cdots$, that is, with the F's of one system, and the v's of the other, in turn,

$$\begin{aligned} 0 &= \tfrac{1}{2}Q_{12} + p_2U_{12} + p_1T_{12}, \\ 0 &= \tfrac{1}{2}Q_{12} + p_1U_{12} + p_2T_{12}; \end{aligned}$$

adding which, there results the equation of mutual activity,

$$0 = Q_{12} + (p_1 + p_2)(U_{12} + T_{12}) \text{ or } 0 = Q_{12} + \dot{U}_{12} + \dot{T}_{12};$$

and, on subtraction, there results

$$0 = (p_1 - p_2)(U_{12} - T_{12}), \tag{91}$$

giving $U_{12} = T_{12}$ if the p's are unequal. But this property is true whether the p's be equal or not; that is, $U_{11} = T_{11}$ when p_1 is a repeated root. I have before discussed various cases of the above, with social reference to the dynamical system expressed by Maxwell's electromagnetic equations.

The above quotation gives Heaviside's approach to the problem. He arrives at the relation of the mutual energies given by (24), Heaviside's equation (91), from the general energy considerations of a dynamical system.

DETERMINATION OF THE AMPLITUDES OF NORMAL DISTRIBUTIONS OF VOLTAGES AND CURRENTS IN A CIRCUIT NETWORK

Assume a network consisting of n circuits, each comprising an inductance and a condenser; energy stored up electrically and magnetically in the circuit system. On removal of the impressed electromotive force, the system will subside to equilibrium in normal current and voltage distributions, the subsidence being represented by the time factor ϵ^{pt}. The voltage and current in any circuit is given by

$$V = \Sigma A u \epsilon^{pt},$$
$$I = \Sigma A w \epsilon^{pt},$$

V being the real voltage at a place where the corresponding normal voltage is u and I the real current where the corresponding normal current is w.

For any particular normal distribution, the amplitudes of the voltage across the n condensers are

$$A_v u_v^{(1)},\ A_v u_v^{(2)},\ \cdots\ A_v u_v^{(n)}$$

where A_v is the coefficient of the normal distribution corresponding to the vth distribution.

The amplitudes of the currents in the inductance coils are

$$A_v w_v^{(1)},\ A_v w_v^{(2)},\ \cdots\ A_v w_v^{(n)}.$$

The effective voltage on any one condenser, of the μth circuit, say, is given by

$$V\mu = A_1 u_1^{(\mu)}\, \epsilon^{p_1 t} + A_2 u_2^{(\mu)}\, \epsilon^{p_2 t} + \cdots \tag{25}$$

The current in the coil of the μ^{th} circuit is given by

$$I_\mu = A_1 w_1^\mu \epsilon^{p_1 t} + A_2 w_2^{(\mu)} \epsilon^{p_2 t} + \cdots \tag{26}$$

Initially, when $t = 0$, the voltage and current in the μ^{th} circuit are given by

$$\left.\begin{aligned} V_{\mu 0} &= A_1 u_1^{(\mu)} + A_2 u_2^{(\mu)} + \cdots \\ I_{\mu 0} &= A_1 w_1^{(\mu)} + A_2 w_2^{(\mu)} + \cdots \end{aligned}\right\} \tag{27}$$

Now calculate the mutual electric and magnetic energies of the given state with respect to, say, the v^{th} normal distribution. To do this, multiply (27) by $C^{(\mu)} u_v^{(\mu)}$ and sum up over all the n circuits. We obtain the following, leaving off the designation μ, and using the summation sign:

$$\begin{aligned} U_{0v} = \Sigma C u_v V_0 &= \Sigma C\{A_1 u_1 u_v + A_2 u_2 u_v + \cdots\} \\ &= A_1 U_{1v} + A_2 U_{2v} + \cdots \end{aligned} \tag{28}$$

The terms U_{k_v} on the right-hand side of the above equation are the mutual electrical energies of any K^{th} distribution and the v^{th} normal distribution.

In a similar way, we obtain the following expression for the magnetic energies:

$$T_{0v} = A_1 T_{1v} + A_2 T_{2v} + \cdots \tag{29}$$

Subtracting (29) from (28), all terms $A_K(U_{kv} - T_{Kv})$ on the right-hand side cancel by the relation (24) except the term $A_v(U_{vv} - T_{vv})$. The energies U_{vv} and T_{vv} are not mutual energies. U_{vv} is double the self energy of the v^{th} normal distribution, and so is $T_{vv} = 2T_v$.

We obtain, therefore, the relation

$$U_{0_v} - T_{0v} = 2A_v(U_v - T_v),$$

and

$$A_v = \frac{U_{0v} - T_{0v}}{2(U_v - T_v)} \tag{30}$$

This relation between the amplitudes of the normal distributions and the energies in the initial state was developed by Heaviside in his paper, "Electromagnetic Induction and its Propagation," collected papers, Vol. I, p. 523.

From this and another relation that he establishes between the energies of the system and the electrical characteristics of the circuit system, the energy terms are eliminated and an expression is obtained for the amplitudes of the normal distributions in terms of the circuit characteristics and the initial states.

The derivation of formula (30) is not set forth in Heaviside's papers connectedly in a way that could be followed easily; hence, the demonstration given above. From this point on, however, the rest of the work in the process of deriving the expansion formula is very clearly given in his papers, and it will be best to quote him directly.[1] The two sections referred to, together with formula (30), give the complete derivation of the expansion formula. Parts of these sections which have specific reference to the problem are given below.

Suppose, for example, we have two fine-wire terminals, a and b, that are joined through any electromagnetic and electrostatic combination which does not contain impressed forces, nor receives energy from without, except by means of the current, say C, entering it at a and leaving it at b. Let also V be the excess of the potential of a over that of b. Then VC is the energy-current or the amount of energy added per second to the combination through the terminal connections with, necessarily, some other combination. . . . The combination need not be of mere linear circuits, in which differences of current-density are insensible; there may, for example, be induction of currents in a mass of matter either connected conductively or not with a and b; but in any case it is necessary that the arrangement should terminate in fine wires at a and b, in order that the two quantities V and C may suffice to specify, by their product, the energy current at the terminals. Even in this we completely ignore the dialectric currents and also the displacement, in the neighborhood of the terminals, *i.e.*, we assume $c = 0$ to stop displacement. This is, of course, what is always done, unless specially allowed for.

Now, supposing the structure of the combination to be given, we can always, by writing out the equations of its different parts, arrive at a characteristic equation connecting the terminal V and C. For instance,

$$V = ZC \tag{98}$$

where Z is a function of d/dt. In the simplest case Z is a mere resistance. A common form of this equation is

$$f_0V + f_1\dot{V} + f_2\ddot{V} + \cdots = g_0C + g_1\dot{C} + g_2\ddot{C} + \cdots$$

where the f's and g's are constants. But there is no restriction to such simple forms. All that is necessary is that the equation should be

[1] It is necessary to refer to only two sections, one given on p. 204, Vol. II, collected papers, under the heading "Applications to Any Electromagnetic Arrangement Subject to $V = ZC$," and the other section on p. 371, Vol. II, collected papers, under the heading "The Use of the Resistance-operator in Normal Solutions."

linear, so that Z may be a function of p. If, for example, $(dC/dt)^2$ occurred, we could not do it.

Now this combination must necessarily be joined on to another, however elementary, to make a complete system, unless V is to be zero always. The complete system, without impressed forces in it, has its proper normal modes of subsidence, corresponding to definite values of p. Consequently

$$U_{12}-T_{12}=(U_2C_1-U_1C_2)\div(p_1-p_2)^1 \tag{99}$$

if V_1, C_1 belong to p_1 and V_2, C_2 to p_2, whilst the left member refers to the combination given by $V = ZC$. Or,

$$U_{12} - T_{12} = C_1C_2\left(\frac{V_1}{C_1} - \frac{V_2}{C_2}\right) \div (p_2 - p_1) = C_1C_2\frac{Z_1 - Z_2}{p_2 - p_1}, \tag{100}$$

and the value of $2(U - T)$ is a single normal system is

$$2(U - T) = V\frac{dC}{dp} - C\frac{dV}{dp} = -C^2\frac{d}{dp}\frac{V}{C} = -C^2\frac{dZ}{dp}. \tag{101}$$

In a similar manner we can write down the energy differences for the complementary combination, whose equation is, say, $V = yC$; remembering that $-VC$ is the energy entering it per second, we get

$$C_1C_2\frac{Y_1 - Y_2}{p_1 - p_2}, \text{ and } C^2\frac{dY}{dp}, \text{ respectively.}$$

By addition, the complete $U_{12} - T_{12}$ is

$$C_1C_2\frac{Y_1 - Y_2 - Z_1 + Z_2}{p_1 - p_2} = 0 = C_1C_2\frac{\varphi_1 - \varphi_2}{p_1 - p_2} \tag{102}$$

and the complete $2(U - T)$ is

$$C^2\frac{d}{dp}(Y - Z) \text{ or } C^2\frac{d\varphi}{dp}, \tag{103}$$

where $\varphi = 0$, or $Y - Z = 0$, is the determinantal equation of the complete system (both combinations which join on at a and b where V and C are reckoned), expressed in such a form that every term in is of the dimensions of a resistance.[2]

THE USE OF THE RESISTANCE-OPERATOR IN NORMAL SOLUTIONS

In conclusion, consider the application of tne resistance-operator to normal solutions. If we leave a combination to itself without impressed force, it will subside to equilibrium (when there is resistance) in a manner determined by the normal distributions of electric and magnetic

[1] See equation (23), p. 149.

[2] Collected papers, Vol. II, pp. 204–206

force, or of charges of condensers and currents in coils; a normal system being, in the most extended sense, a system that, in subsiding, remains similar, the subsidence being represented by the time factor ϵ^{pt}, where p is a root of the equation $Z = 0$. It is true that each part of the combination will usually have a distinct resistance-operator; but the resistance-operators of all parts involve, and are contained in, the same characteristic function, which is merely the Z of any part cleared of fractions. It is sometimes useful to remember that we should clear of fractions, for the omission to do so may lead to the neglect of the whole series of roots; but such cases are exceptional and may be foreseen; while the employment of a resistance-operator rather than the characteristic function is of far greater general utility, both for ease of manipulation and for physical interpretation.

Given a combination containing energy and left to itself, it is upon the distribution of the energy that the manner of subsidence depends, or upon the distribution of the electric and magnetic forces in those parts of the system where the permittivity and the inductivity are finite, or are reckoned finite for the purpose of calculation. Thus conductors, if they be not also dielectrics, have only to be considered as regards the magnetic force, whilst in a dielectric, we must consider both the electric and the magnetic force. Now the internal connexions of the system determine what ratios the variables chosen should bear to one another in passing from place to place in order that the resulting system should be known; and a constant multiplier will fix the size of the normal system. Thus, supposing u and w are the normal functions of voltage and current, which are in most problems the most practical variables, the state of the whole system at time t will be represented by

$$V = \Sigma A u \epsilon^{pt}, \ C = \Sigma A w \epsilon^{pt}; \tag{46}$$

V being the real voltage at a place where the corresponding normal voltage is u, and C the real current where the normal current is w, the summation extending over all the p-roots of the characteristic equation. The size of the systems, settled by the A's (one for each p) are to be found by the conjugate property of the vanishing of the mutual energy difference of any pair of p-systems, applied to the initial distributions of V and C.

To find the effect of impressed force is a frequently recurring problem in practical applications; and here the resistance-operator is specially useful, giving a general solution of great simplicity. Thus, suppose we insert a steady impressed force e at a place where the resistance-operator is Z, producing $e = ZC$ thereafter. Find C in terms of e and Z. The following demonstration appears quite comprehensive. Convert the problem into a case of subsidence first, by substituting a condenser of permittance S, and initial charge Se, for the impressed force. By

making S infinite later we arrive at the effect of the steady e. In getting the subsidence solution we have only to deal with the energy of the condenser, so that a knowledge of the internal connexions of the system is quite superfluous.

The resistance-operator of the condenser being $(Sp)^{-1}$, that of the combination, when we use the condenser, is Z_1, where

$$Z_1 = (Sp)^{-1} + Z. \tag{47}$$

Let V and C be the voltage and the current respectively at time t after insertion of the condenser, and due entirely to its initial charge. Equations (46) above express them, if u and w have the special ratio proper at the condenser, given by

$$w = -Spu \tag{48}$$

because the current equals the rate of decrease of its charge. Initially, we have $e = \Sigma Au$ and $\Sigma Aw = 0$. So, making use of the conjugate property, we have

$$Seu = 2(U_p - T_p)A_1[1] \tag{49}$$

if U_p be the electric and T_p the magnetic energy in the normal system. But the following property of the resistance-operator is also true.

$$2(T_p - U_p) = \frac{dZ_1}{dp}w^2;[2] \tag{50}$$

That is, dZ_1/dp is the impulsive inductance in the p-system at a place where the resistance-operator is Z_1, p being a root of $Z_1 = 0$; just as dZ_1/dp with $p = 0$ is the impulsive inductance (complete) at the same place. Using (50) and (49) gives

$$A = -(Seu) \div \left(w^2\frac{dZ_1}{dp}\right). \tag{51}$$

Now use (48) in (51) and insert the resulting A in the second of (46), and there results

$$C = \sum \frac{e}{pZ_1'}\epsilon^{pt} \tag{52}$$

where the accent means differentiation to p. This is the complete subsidence solution. Now increase S infinitely, keeping e constant.

Z_1 ultimately becomes Z; but in doing so one root of $Z_1 = 0$ becomes zero. We have, by (47), and remembering that $Z_1 = 0$

$$pZ_1' = -(Sp)^{-1} + pZ' = Z + pZ'; \tag{53}$$

[1] See equation (30), p. 152; the initial energy u_{0_u} condenser charge no initial magnetic energy.

[2] See equation (103) p. 154.

so, when $S = \infty$ and $Z = 0$, we have $pZ_1' = pZ'$ for all roots except the one just mentioned, in which case p tends to zero and Z' is finite, making in the limit $pZ_1 = Z_0$ by (53), where Z_0 is the $p = 0$ value of Z, or the steady resistance. Therefore, finally,

$$C = \frac{e}{Z_0} + \sum \frac{e}{pZ'}\epsilon^{pt}, \tag{54}$$

where the summation extends over the roots of $Z = 0$, shows the manner of establishment of the current by the impressed force e. The use of this equation (54), even in comparatively elementary problems, leads to a considerable saving of labor whilst in cases involving partial differential equations, it is invaluable.[1]

It is interesting to compare the methods of deriving the expansion formula as originally given by Heaviside, and the method developed in Chap. II. It is observed that Heaviside starts from certain fundamental considerations of energy distributions in an electric-circuit system, first arriving at an expression for the energy subsidence when the system is left to itself, no external electromotive forces acting on it, and then developing the more general expression for the current rise in an electric-circuit system under the action of an impressed electromotive force. In the method given in Chap. II, the subject is discussed from the standpoint of current distributions in a circuit system, arriving at the general-expression formula for the current rise in a circuit system under the application of an electromotive force, and from this the formula for current subsidence in a circuit system is deduced.

[1] Collected papers, Vol. II, pp. 37–373.

APPENDIX

NOTE ON BESSEL FUNCTIONS

The Bessel functions of the first kind $J_n(x)$ and $I_n(x)$ are defined, when n is zero or a positive integer, by the absolutely convergent series:

$$J_n(x) = \frac{x^n}{2^n(n!)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \frac{x^6}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)} + \cdots\right\}$$

$$I_n(x) = \frac{x^n}{2^n(n!)}\left\{1 + \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} + \frac{x^6}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)} + \cdots\right\}$$

The function $J_n(x)$ satisfies the differential equation

$$\frac{d^2J_n(x)}{dx^2} + \frac{1}{x}\frac{dJ_n(x)}{dx} + \left(1 - \frac{n^2}{x^2}\right)J_n(x) = 0,$$

and the function $I_n(x)$ satisfies the differential equation

$$\frac{d^2I_n(x)}{dx^2} + \frac{1}{x}\frac{dI_n(x)}{dx} - \left(1 + \frac{n^2}{x^2}\right)I_n(x) = 0.$$

The connection between the J and I functions is given by the following equations:

$$I_n(x) = j^{-n}J_n(x),$$

$$(j = \sqrt{-1})$$

$$I_{-n}(x) = j^nI_{-n}(x).$$

The series given above $J_n(x)$ and $I_n(x)$ become practically useless for numerical computations when the argument x is even moderately large. These functions, however, are expressible in

other series forms which are well adapted for numerical calculations for large values of the argument. Thus:

$$I_n(x) = \frac{\epsilon^x}{\sqrt{2\pi x}}\left\{1 - \frac{4n^2-1}{1!8x} + \frac{(4n^2-1)(4n^2-9)}{2!(8x)^2} \right.$$
$$\left. \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8x)^3} + \cdots \right\}$$

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left\{P_n \cos\left(x - \frac{2n+1}{4}\pi\right) - Q_n \sin\left(x - \frac{2n+1}{4}\pi\right)\right\}$$

where

$$P_n = 1 - \frac{(4n^2-1)(4n^2-9)}{2!(8x)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!(8x)^4} - \cdots$$

$$Q_n = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8x)^3} + \cdots$$

For very large values of x, the functions approximate the values given by

$$I_n(x) = \frac{\epsilon^x}{\sqrt{2\pi x}},$$

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2n+1}{4}\pi\right).$$

The following formulas are useful giving relations between the functions and their differentials, I'_n and J'_n indicating the differentials of the functions:

$$J'_n = \frac{n}{x}J_n - J_{n+1}$$

$$J'_n = -\frac{n}{x}J_n + J_{n-1}$$

$$J_{n+1} - \frac{2n}{x}J_n + J_{n-1} = 0$$

$$J'_0 = -J_1$$

Similarly, for the I functions,

$$I'_n = \frac{n}{x}I_n + I_{n+1}$$

$$I'_n = -\frac{n}{x}I_n + I_{n-1}$$

$$I_{n+1} + \frac{2n}{x}I_n + I_{n-1} = 0$$

$$I'_0 = I_1$$

Derivation of formula,

$$(a^2 - p^2)\frac{I_m(at)}{(at)^m} = (2m+1)\frac{p}{t}\frac{I_m(at)}{(at)^m}$$
$$= (2m+1)a^2\frac{I_{m+1}(at)}{(at)^{m+1}}.$$

This formula is used in Chap. V in the solution of wave propagation problems (see p. 104).

Disregard the constant factor a for the moment. We have, by direct differentiation,

$$p\frac{I_m(t)}{t^m} = \frac{t^m pI_m(t) - mt^{m-1}I_m(t)}{t^{2m}} = \frac{pI_m(t)}{t^m} - \frac{mI_m(t)}{t^{m+1}}.$$

By the relation given above, we have

$$pI_m(t) = \frac{m}{t}I_m(t) + I_{m+1}(t).$$

Hence,

$$p\,\frac{I_m(t)}{t^m} = \frac{I_{m+1}(t)}{t^m} = -\frac{2m}{t^{m+1}}\,I_{m+1}(t) + \frac{I_{m-1}(t)}{t^m}.$$

Differentiating again,

$$p^2\frac{I_m(t)}{t^m} = p\left\{\frac{-2m}{t^{m+1}}\,I_m(t) + \frac{I_{m-1}(t)}{t^m}\right\}$$
$$= \frac{-2m}{t}p\frac{I_m(t)}{(t)^m} + \frac{2m}{t^2}\frac{I_m(t)}{t^m} + \frac{1}{t^m}pI_{m-1}(t) - \frac{m}{(t)^{m+1}}I_{m-1}(t)$$
$$= \frac{-2m}{t}p\frac{I_m(t)}{(t)^m} + \frac{2m}{t^2}\frac{I_m(t)}{(t)^m} + \frac{1}{t^m}\left(\frac{m-1}{t}I_{m-1}(t) + I_m(t)\right) - \frac{m}{t}\frac{I_{m-1}(t)}{t^m}$$
$$= \frac{-2m}{t}p\frac{I_m(t)}{t^m} + \frac{2m}{t^2}\frac{I_m(t)}{t^m} - \frac{1}{t}\frac{I_{m-1}(t)}{t^m} + \frac{I_m(t)}{t^m}$$
$$= \frac{-2m}{t}p\frac{I_m(t)}{t^m} + \frac{2m}{t^2}\frac{I_m(t)}{t^m} - \frac{1}{t}p\frac{I_m(t)}{t^m} - \frac{2m}{t^2}\frac{I_m}{t^m} + \frac{I_m(t)}{t^m},$$

and

$$(p^2 - 1)\frac{I_m(t)}{t^m} = \frac{-(2m+1)}{t}p\frac{I_m(t)}{t^m}$$
$$= (2m+1)\frac{I_{m+1}(t)}{t^{m+1}}.$$

If we replace t by at, we have

$$p = \frac{d}{d(at)} = \frac{1}{a}\frac{d}{dt}$$

$$p^2 = \frac{1}{a^2}\frac{d^2}{dt^2},$$

and the above transforms to

$$\left(\frac{p^2}{a^2} - 1\right)\frac{I_m(at)}{(at)^m} = (2m + 1)\frac{I_{m+1}(at)}{(at)^{m+1}},$$

or

$$(p^2 - a^2)\frac{I_m(a^t)}{(at)^m} = (2m + 1)a^2\frac{I_{m+1}(a^t)}{(at)^{m+1}}.$$

MATHEMATICAL FORMULAS

For convenience, the more frequently used mathematical formulas in the text are assembled here.

$$\epsilon^x = \cosh x + \sinh x$$

$$\epsilon^{-x} = \cosh x - \sinh x$$

$$\epsilon^{jx} = \cos x + j \sin x$$

$$\epsilon^{-jx} = \cos x - j \sin x$$

$$\cos x = \cosh jx; \cosh x = \cos jx$$

$$\sin x = -j \sinh jx; \sinh x = -j \sin jx$$

$$\tan x = -j \tanh x; \tanh x = -j \tan jx$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sin \frac{x}{2} = \sqrt{\frac{1}{2}(1 - \cos x)}; \cos \frac{x}{2} = \sqrt{\frac{1}{2}(1 + \cos x)}.$$

$$\sinh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x - 1)}; \cosh \frac{x}{2} = \sqrt{\frac{1}{2}(\cosh x + 1)}.$$

$$\sin (x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos (x \pm y) = \cos x \cos y \pm \sin x \sin y$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh (mj\pi) = 0 \; (m \text{ is an integer})$$

$$\cosh (mj\pi) = (-1)^m$$

$$\tanh (mj\pi) = 0$$

$$\sinh (x + mj\pi) = (-1)^m \sinh x$$

$$\cosh (x + mj\pi) = (-1)^m \cosh x$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\sinh 3x = 4 \sinh^3 x + 3 \sinh x = \sinh x \,(4 \cosh^2 x - 1)$$

$$\sinh (n+1)x = 2 \cosh x \sinh nx - \sinh (n-1)x$$

$$\sinh nx = n \cosh^{n-1} x \sinh x + \frac{n(n-1)(n-2)}{6} \cosh^{n-3} x \sinh^3 x + \cdots$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x = \cosh x(4 \sinh^2 x + 1)$$

$$\cosh (n+1)x = 2 \cosh x \cosh nx - \cosh (n-1)x$$

$$\cosh nx = \cosh^n x + \frac{n(n-1)}{2} \cosh^{n-2} x \sinh^2 x + \cdots$$

$$\epsilon^{\pm x} = 1 \pm \frac{x}{1!} + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{x}{\sinh x} = 1 - \frac{2}{1 + \left(\frac{\pi}{x}\right)^2} + \frac{2}{1 + \left(\frac{2\pi}{x}\right)^2} - \frac{2}{1 + \left(\frac{3\pi}{x}\right)^2} + \cdots$$

$$x \coth x = 1 + \frac{2}{1 + \left(\frac{\pi}{x}\right)^2} + \frac{2}{1 + \left(\frac{2\pi}{x}\right)^2} + \frac{2}{1 + \left(\frac{3\pi}{x}\right)^2} + \cdots$$

$$\sin x = x \prod_1^\infty \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

$$\cos x = \prod_1^\infty \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right)$$

$$\cot x = \frac{1}{x} + 2 \sum_1^\infty \frac{x}{x^2 - n^2\pi^2}$$

$$\tan x = 2 \sum_1^\infty \frac{x}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - x^2}$$

$$\sec x = \sum_{1}^{\infty} \frac{(-1)^{n-1}(2n-1)\pi}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - x^2}$$

$$\operatorname{cose} x = \frac{1}{x} + \sum_{1}^{\infty} \frac{(-1)^{n-1} 2x}{n^2\pi^2 - x^2}$$

$$(1 + x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m - 1(m-2)x^3}{3!} + \cdots$$

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^5 + \cdots$$

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^5 - \cdots$$

TABLE I.—BESSEL FUNCTIONS

x	$J_0(x)$	$J_1(x)$	$I_0(x)$	$I_1(x)$
0.0	1.00000	0.00000	1.00000	0.00000
0.2	0.99002	0.09950	1.01002	0.10050
0.4	0.96040	0.19603	1.04040	0.20403
0.6	0.91200	0.28670	1.09205	0.31370
0.8	0.84629	0.36884	1.16651	0.43286
1.0	0.76520	0.44005	1.26607	0.56516
1.2	0.67113	0.49829	1.39373	0.71468
1.4	0.56685	0.54195	1.55340	0.88609
1.6	0.45540	0.56990	1.74998	1.08481
1.8	0.33999	0.58152	1.98956	1.31717
2.0	0.22389	0.57672	2.27959	1.59064
2.2	0.11036	0.55596	2.62914	1.91409
2.4	0.00251	0.52019	3.04926	2.29812
2.6	−0.09680	0.47082	3.55327	2.75538
2.8	−0.18504	0.40970	4.15730	3.30106
3.0	−0.26005	0.33906	4.88079	3.95337
3.2	−0.32019	0.26134	5.74721	4.73425
3.4	−0.36430	0.17923	6.78481	5.67010
3.6	−0.39177	0.09547	8.02768	6.79271
3.8	−0.40256	0.01282	9.51689	8.14042
4.0	−0.39715	−0.06604	11.30192	9.75947
4.2	−0.37656	−0.13865	13.44246	11.70562
4.4	−0.34226	−0.20278	16.01043	14.04622
4.6	−0.29614	−0.25655	19.09262	16.86256
4.8	−0.24043	−0.29850	22.79368	20.25283
5.0	−0.17760	−0.32758	27.23987	24.33564
5.2	−0.11029	−0.34322	32.58359	29.25431
5.4	−0.04121	−0.34534	39.00879	35.18206
5.6	+0.02697	−0.33433	46.73755	42.32829
5.8	+0.09170	−0.31103	56 03810	50.94618
6.0	+0.15064	−0.27668	67.23441	61.34194

TABLE II.—EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	ϵ^x	ϵ^{-x}	Cosh x	Sinh x
0.0	1.0000	1.0000	1.0000	0.0000
0.2	1.2214	0.8187	1.02007	0.20134
0.4	1.4918	0.6703	1.08107	0.41075
0.6	1.8221	0.5488	1.18547	0.63665
0.8	2.2225	0.4493	1.33743	0.88811
1.0	2.7183	0.3679	1.54308	1.17520
1.2	3.3201	0.3012	1.81066	1.50946
1.4	4.0552	0.2466	2.15090	1.90430
1.6	4.9530	0.2019	2.57746	2.37557
1.8	6.0496	0.1653	3.10747	2.94217
2.0	7.3891	0.1353	3.76220	3.62686
2.2	9.0250	0.1108	4.56791	4.45711
2.4	11.0232	0.0907	5.55695	5.46623
2.6	13.4637	0.07427	6.76900	6.69473
2.8	16.4446	0.06081	8.25273	8.19192
3.0	20.0855	0.04979	10.0677	10.0179
3.2	24.5325	0.04076	12.2866	12.2459
3.4	29.9641	0.03337	14.9987	14.9654
3.6	36.5982	0.02732	18.3128	18.2855
3.8	44.7012	0.02237	22.3618	22.3394
4.0	54.5982	0.01832	27.3082	27.2899
4.2	66.6863	0.01500	33.3507	33.3357
4.4	81.4509	0.01228	40.7316	40.7193
4.6	99.4843	0.01005	49.7472	49.7371
4.8	121.510	0.00823	60.7593	60.7511
5.0	148.413	0.00674	74.2099	74.2032
5.2	181.272	0.00552	90.6388	90.6333
5.4	221.406	0.00452	110.705	110.701
5.6	270.426	0.00370	135.215	135.211
5.8	330.300	0.00303	165.151	165.148
6.0	403.429	0.00248	201.716	201.713

TABLE III.—$\epsilon rf y = \frac{2}{\sqrt{\pi}} \int_0^y \epsilon^{-y^2} dy$

y	$\epsilon rf y$	Δ	y	$\epsilon rf y$	Δ
0.5	0.05637	0.05637	1.05	0.86243	0.01973
0.10	0.11246	0.05609	1.10	0.88020	0.01777
0.15	0.16799	0.05553	1.15	0.89612	0.01592
0.20	0.22270	0.05471	1.20	0.91031	0.01419
0.25	0.27632	0.05362	1.25	0.92290	0.01259
0.30	0.32862	0.05230	1.30	0.93400	0.01110
0.35	0.37938	0.05076	1.35	0.94376	0.00976
0.40	0.42839	0.04901	1.40	0.95228	0.00852
0.45	0.47548	0.04709	1.45	0.95969	0.00741
0.50	0.52049	0.04501	1.50	0.96610	0.00641
0.55	0.56332	0.04283	1.55	0.97162	0.00552
0.60	0.60385	0.04053	1.60	0.97634	0.00472
0.65	0.64202	0.03817	1.65	0.98037	0.00403
0.70	0.67780	0.03578	1.70	0.98379	0.00342
0.75	0.71115	0.03335	1.75	0.98667	0.00288
0.80	0.74210	0.03095	1.80	0.98909	0.00242
0.85	0.77066	0.02856	1.85	0.99111	0.00202
0.90	0.79690	0.02624	1.90	0.99279	0.00168
0.95	0.82089	0.02399	1.95	0.99417	0.00138
1.00	0.84270	0.02181	2.00	0.99532	0.00115

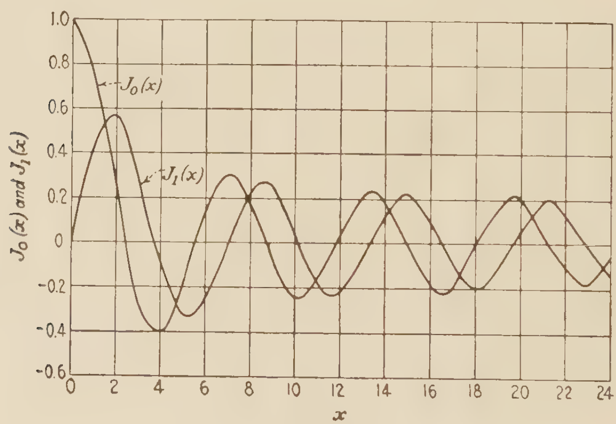

FIG. 28.

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